

# MULTIPLE COMMUTATOR FORMULAS FOR UNITARY GROUPS

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**ABSTRACT.** Let  $(A, \Lambda)$  be a form ring such that  $A$  is quasi-finite  $R$ -algebra (i.e., a direct limit of module finite algebras) with identity. We consider the hyperbolic Bak's unitary groups  $\mathrm{GU}(2n, A, \Lambda)$ ,  $n \geq 3$ . For a form ideal  $(I, \Gamma)$  of the form ring  $(A, \Lambda)$  we denote by  $\mathrm{EU}(2n, I, \Gamma)$  and  $\mathrm{GU}(2n, I, \Gamma)$  the relative elementary group and the principal congruence subgroup of level  $(I, \Gamma)$ , respectively. Now, let  $(I_i, \Gamma_i)$ ,  $i = 0, \dots, m$ , be form ideals of the form ring  $(A, \Lambda)$ . The main result of the present paper is the following multiple commutator formula

$$\begin{aligned} [\mathrm{EU}(2n, I_0, \Gamma_0), \mathrm{GU}(2n, I_1, \Gamma_1), \mathrm{GU}(2n, I_2, \Gamma_2), \dots, \mathrm{GU}(2n, I_m, \Gamma_m)] = \\ [\mathrm{EU}(2n, I_0, \Gamma_0), \mathrm{EU}(2n, I_1, \Gamma_1), \mathrm{EU}(2n, I_2, \Gamma_2), \dots, \mathrm{EU}(2n, I_m, \Gamma_m)], \end{aligned}$$

which is a broad generalization of the standard commutator formulas. This result contains all previous results on commutator formulas for classical like-groups over commutative and finite-dimensional rings.

## 1. INTRODUCTION

Let  $(A, \Lambda)$  be a form ring,  $n \geq 3$  and let  $\mathrm{GU}(2n, A, \Lambda)$  be the hyperbolic Bak's unitary group [19, 33, 9, 28]. In the paper [29] we obtained relative commutator formulas for the unitary groups  $\mathrm{GU}(2n, A, \Lambda)$ , under some natural commutativity/finiteness assumptions on  $(A, \Lambda)$ . The goal of the present paper is to enhance the relative localisation method developed in [29] and to prove *multiple* relative commutator formulas, which serve as a simultaneous generalisation of all previously known such results. For the general linear group  $\mathrm{GL}(n, A)$  similar results were recently established by the first and the third author in [32].

Actually, since the general linear group is a special case of Bak's unitary group, the results of the present paper are not only modeled on [32] but also generalise the results of [32]. Our results are new already in the following classical situations.

- The case of symplectic groups  $\mathrm{Sp}(2l, R)$ , where the involution is trivial, and  $\Lambda = R$ .
- The case of even split orthogonal groups  $\mathrm{SO}(2l, R)$ , where the involution is trivial and  $\Lambda = 0$ .

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- The case of classical unitary groups  $SU(2l, R)$ , where  $\Lambda = \Lambda_{\max}$ .

As the proofs in [32], the proofs in the present paper are based on a version of localisation. The two most familiar versions of localisation are Quillen–Suslin’s localisation and patching, and Bak’s localisation-completion. Actually, in this paper we use a version of Bak’s method [3], which was first applied to unitary groups in the Bielefeld Thesis of the first author [20, 21]. It is interesting to note that in this generality the first convincing treatment of Quillen–Suslin’s localisation and patching method appeared only afterwards, in Petrov’s Saint Petersburg Thesis [41, 42, 43]. We do not attempt to give an account of the historical development of localisation methods, see our surveys [28, 22] for more details and many related references.

More precisely, our proofs rely on a further enhancement of the *relative* localisation method introduced by the first and the third author [31] in the context of the general linear group, and applied to Bak’s unitary groups in [29] and to Chevalley groups in [30]. Initially, this method was proposed to address problems raised by Alexei Stepanov and the second author [59]. See our published papers [27, 5, 31, 29, 49, 32] and our forthcoming papers [30, 23, 24, 25, 26] for many further recent applications of this method and other offsprings of Bak’s method, including the remarkable universal localisation by Alexei Stepanov [47]. Compare also the recent papers by Anthony Bak, Rabeya Basu, Khanna, Alexander Luzgarev, Victor Petrov, Ravi Rao, Anastasia Stavrova and Matthias Wendt [41, 42, 15, 13, 4, 44, 14, 61, 36, 46], for latest versions and fresh applications of Quillen–Suslin’s method.

Since the present paper is an immediate sequel of [32, 29], we do not reproduce a detailed historical survey of the commutator formulas, and do not discuss crucial early contributions by Hyman Bass [10, 11], Anthony Bak [1, 12], Andrei Suslin and Vyacheslav Kopeiko [50, 51, 34, 54], Alec Mason and Wilson Stothers [40, 37, 38, 39], Leonid Vaserstein [55, 56], Zenon Borewicz and the second author [16], Giovanni Taddei [52], and others. Instead, we refer to our surveys [58, 28] and to the papers [9, 48] for an accurate historic description and many further references.

However, to put our results in context, let us briefly review the standard commutator formulas for unitary groups. Below,  $EU(2n, A, \Lambda)$  denotes the [absolute] elementary unitary group, generated by the elementary root unipotents. As usual, for a form ideal  $(I, \Gamma)$  of the form ring  $(A, \Lambda)$  we denote by  $EU(2n, I, \Gamma)$  the corresponding relative elementary subgroup, by  $GU(2n, I, \Gamma)$  the principal congruence subgroup of level  $(I, \Gamma)$  and by  $CU(2n, I, \Gamma)$  the full congruence subgroup of level  $(I, \Gamma)$ .

One of the main results of Bak’s Thesis [1], Theorem 1.2 can be summarised as follows. Actually, the second formula is not part of the statement of that theorem, but it appears in its proof, at the bottom of page 4.2, see corollary 3.4 on page 3.22 of [1], or [12]. The group  $CU(2n, I, \Gamma)$  is defined differently, but from [9] we know that in all interesting situations, including the ones covered by Theorems 1 and 2 below, all definitions of these groups coincide.

**Theorem 1** (Bak). *Let  $R$  be a Noetherian commutative ring of Bass–Serre dimension  $d$  and let  $(A, \Lambda)$  be a form ring module finite over  $R$ -algebra. Assume that  $n \geq d + 1, 3$ . Further, let  $(I, \Gamma)$  be a form ideal of the form ring  $(A, \Lambda)$ . Then*

$$[GU(2n, A, \Lambda), EU(2n, I, \Gamma)] = [EU(2n, A, \Lambda), CU(2n, I, \Gamma)] = EU(2n, I, \Gamma). \quad (1)$$

The following result is referred to as the *absolute standard commutator formula*. In this generality it is established by Anthony Bak and the second author [8, 9] and by Leonid Vaserstein and You Hong [57].<sup>1</sup> This result, and the more general Theorem 4 will be used throughout the present paper.

**Theorem 2** (Bak–Vavilov, Vaserstein–You Hong). *Let  $n \geq 3$ ,  $R$  be a commutative ring,  $(A, \Lambda)$  be a form ring such that  $A$  is a quasi-finite  $R$ -algebra. Further, let  $(I, \Gamma)$  be a form ideal of the form ring  $(A, \Lambda)$ . Then*

$$[\mathrm{GU}(2n, A, \Lambda), \mathrm{EU}(2n, I, \Gamma)] = [\mathrm{EU}(2n, A, \Lambda), \mathrm{CU}(2n, I, \Gamma)] = \mathrm{EU}(2n, I, \Gamma). \quad (2)$$

In the context of Bak’s unitary groups, the history of relative standard commutator formula starts with the Thesis of Günter Habdank [17], see also [18]. The following result is essentially [18], Proposition 3.5. Actually, there it is stated in terms of a certain rather technical quadratic stable rank condition. Not to recall its definition here, we limit ourselves with a special case of this result, under the same assumption as Bak’s theorem.

**Theorem 3** (Habdank). *Let  $R$  be a Noetherian commutative ring of Bass–Serre dimension  $d$  and let  $(A, \Lambda)$  be a form ring module finite over  $R$ -algebra. Assume that  $n \geq d + 1, 3$ . Further, let  $(I, \Gamma)$  and  $(J, \Delta)$  be two form ideals of the form ring  $(A, \Lambda)$ . Then*

$$[\mathrm{EU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)] = [\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)]. \quad (3)$$

For unitary groups, the following result was proven in our previous paper [29]. Both this result itself, and the methods used in its proof are instrumental throughout the present paper. Actually, it is the induction base of our main theorem, and will be repeatedly invoked in its proof.

It is modeled on early contributions by Alec Mason and Wilson Stothers [38, 37, 39, 40]. For the general linear group we gave *three* independent proofs of a similar result: Stepanov–Vavilov [59], based on decomposition of unipotents, Hazrat–Zhang [31], based on localisation, and Stepanov–Vavilov [60], based on the absolute commutator formula and level calculations. Unfortunately – or, maybe, fortunately! – at that time we were not aware of the *extremely* important paper by You Hong [62], where a similar result was obtained for Chevalley groups, with a proof very close to the *second* proof by Stepanov and the second author, [60]. For otherwise we would not be as eager to develop a localisation proof.

**Theorem 4** (Hazrat–Vavilov–Zhang). *Let  $n \geq 3$ ,  $R$  be a commutative ring,  $(A, \Lambda)$  be a form ring such that  $A$  is a quasi-finite  $R$ -algebra. Further, let  $(I, \Gamma)$  and  $(J, \Delta)$  be two form ideals of the form ring  $(A, \Lambda)$ . Then*

$$[\mathrm{EU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)] = [\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)]. \quad (4)$$

Such was the state of art before the present paper. Here, we generalise all these formulas to an arbitrary number of form ideals. The main result of the present paper may be stated as follows. For a start, multiple commutators can be interpreted as left normed commutators.

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<sup>1</sup>The paper [57] uses a naive form of reduction modulo a form ideal, instead of the correct form proposed in [1], see also [9]. Thus, strictly speaking, the proofs in [57] are only valid when  $\Lambda = \Lambda_{\max}$ . Still, it is independent of [8] and at the time was a non-trivial contribution to our understanding of the structure of unitary groups.

**Theorem 5.** *Let  $n \geq 3$ ,  $R$  be a commutative ring,  $(A, \Lambda)$  be a form ring such that  $A$  is a quasi-finite  $R$ -algebra. Furthermore, let  $(I_i, \Gamma_i)$ ,  $i = 0, \dots, m$ , be form ideals of the form ring  $(A, \Lambda)$ . Then*

$$\begin{aligned} & [\text{EU}(2n, I_0, \Gamma_0), \text{GU}(2n, I_1, \Gamma_1), \text{GU}(2n, I_2, \Gamma_2), \dots, \text{GU}(2n, I_m, \Gamma_m)] = \\ & = [\text{EU}(2n, I_0, \Gamma_0), \text{EU}(2n, I_1, \Gamma_1), \text{EU}(2n, I_2, \Gamma_2), \dots, \text{EU}(2n, I_m, \Gamma_m)]. \end{aligned}$$

This result is interesting in itself, but its true significance is that it is absolutely indispensable to proceed to the proof of the *general* multiple commutator formula, which simultaneously generalises both the standard commutator formulas and the nilpotent structure of relative  $K_1$  established in [20, 21, 5]. Using a whole bunch of difficult external results the authors and Alexei Stepanov [25] have been able to establish such a general multiple commutator formula for the case of  $\text{GL}(n, R)$ , but for other groups many tools are still missing, and Theorem 5 bridges one of these gaps.

Multiple formula is also relevant as a prerequisite for the description of subnormal subgroups of unitary groups. See §11 for further comments on these and other possible applications.

Actually, in §10 we prove a still more general result, where both the position of the elementary factor in the left hand side, and the arrangement of brackets may be arbitrary. In fact, Theorem 5 almost immediately implies the following result.

**Theorem 6.** *Let  $n \geq 3$ ,  $R$  be a commutative ring,  $(A, \Lambda)$  be a form ring such that  $A$  is a quasi-finite  $R$ -algebra. Furthermore, let  $(I_i, \Gamma_i)$ ,  $i = 0, \dots, m$ , be form ideals of the form ring  $(A, \Lambda)$  and  $G_i$  be subgroups of  $\text{GU}(2n, A, \Lambda)$  such that*

$$\text{EU}(2n, I_i, \Gamma_i) \subseteq G_i \subseteq \text{GU}(2n, I_i, \Gamma_i), \quad \text{for } i = 0, \dots, m.$$

*If there is an index  $j$  such that  $G_j = \text{EU}(2n, I_j, \Gamma_j)$ , then*

$$[[G_0, G_1, \dots, G_m]] = [[E(I_0), E(I_1), \dots, E(I_m)]]. \quad (5)$$

Observe though, that the arrangement of brackets in (5) should be the same on both sides, the mixed commutators are *not* associative! In particular, there are easy counter-examples which show that in general

$$\begin{aligned} & [[\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)], \text{EU}(2n, K, \Omega)] \neq \\ & [\text{EU}(2n, I, \Gamma), [\text{EU}(2n, J, \Delta), \text{EU}(2n, K, \Omega)]]. \end{aligned}$$

Actually, the difficult part of the proof of Theorem 5, which allows to carry through inductive step is the following *triple* commutator formula. It is precisely the proof of that special case that requires new ideas, as compared with [29], and entails most of the technical strain.

**Theorem 7.** *Let  $n \geq 3$ ,  $R$  be a commutative ring,  $(A, \Lambda)$  be a form ring such that  $A$  is a quasi-finite  $R$ -algebra. Further, let  $(I, \Gamma)$ ,  $(J, \Delta)$  and  $(K, \Omega)$  be three form ideals of a form ring  $(A, \Lambda)$ . Then*

$$\begin{aligned} & [[\text{EU}(2n, I, \Gamma), \text{GU}(2n, J, \Delta)], \text{GU}(2n, K, \Omega)] = \\ & [[\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)], \text{EU}(2n, K, \Omega)]. \end{aligned}$$

In turn, modulo the standard commutator formula the proof of Theorem 7 amounts to the proof of the following equality:

$$\begin{aligned} \left[ \left[ \text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta) \right], \text{GU}(2n, K, \Omega) \right] = \\ \left[ \left[ \text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta) \right], \text{EU}(2n, K, \Omega) \right]. \end{aligned}$$

Essentially, the proof of this last equality is the technical core of the present paper. It cannot be established with the use of the relative commutator calculus developed in our paper [29]. In fact, to prove it we have to develop another layer of the commutator calculus, which works with the generators of  $[\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)]$  rather than with the usual elementary generators.

Let us mention an *amazing* corollary of our results. It shows that *any* multiple commutator of relative elementary groups always equals a *double* mixed commutator, for some other form ideals. In the following lemma  $(I, \Gamma) \circ (J, \Delta)$  denotes the symmetrised product of form ideals, whose definition is recalled in §2. This product is not associative, the bracketing of the form ideals on the right hand side should correspond to the bracketing of commutators on the left-hand side.

**Theorem 8.** *Let  $n \geq 3$ ,  $R$  be a commutative ring,  $(A, \Lambda)$  be a form ring such that  $A$  is a quasi-finite  $R$ -algebra. Furthermore, let  $(I_i, \Gamma_i)$ ,  $i = 0, \dots, m$ , be form ideals of the form ring  $(A, \Lambda)$ . Consider an arbitrary configuration of brackets  $[[\dots]]$  and assume that the outermost pair of brackets between positions  $k$  and  $k + 1$ . Then*

$$\begin{aligned} \llbracket \text{EU}(2n, I_0, \Gamma_0), \text{EU}(2n, I_1, \Gamma_1), \text{EU}(2n, I_2, \Gamma_2), \dots, \text{EU}(2n, I_m, \Gamma_m) \rrbracket = \\ = \left[ \text{EU}(2n, (I_0, \Gamma_0) \circ \dots \circ (I_k, \Gamma_k)), \text{EU}(2n, (I_{k+1}, \Gamma_{k+1}) \circ \dots \circ (I_m, \Gamma_m)) \right]. \end{aligned}$$

As opposed to that, in general the double commutators  $[\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)]$  do not coincide with the elementary subgroups  $\text{EU}(2n, (I, \Gamma) \circ (J, \Delta))$ . This is indeed the case, when  $I$  and  $J$  are comaximal,  $I + J = A$ , but without this additional assumption there are counter-examples even for such nice rings as Dedekind rings of arithmetic type.

The paper is organised as follows. In §§2–4 we recall basic notation, and some background facts concerning form rings and form ideals, Bak’s unitary groups and their relative subgroups, on which the rest of the paper relies. The rest of the paper is devoted to detailed proofs of the above Theorems 5 and 6. In §5 and §6 we prove two important general results of technical nature, which improve and elaborate the results of [9] and [29]. Namely, in §5 we finalise the calculation of levels for the mixed commutator subgroups  $[\text{GU}(2n, I, \Gamma), \text{GU}(2n, J, \Delta)]$ , whereas in §6 we construct a generating system of the mixed elementary commutator subgroups  $[\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)]$ . The next three sections constitute the technical core of the paper. Namely, in §7 and §8 we unfold another layer of the relative commutator calculus, with the elementary generators being replaced by our new generators of the mixed commutator subgroups. This brings us to the stage, where we can carry through the usual patching procedure to prove the *triple* relative commutator formula. This is accomplished in §9. At this point, we are almost there: the rest follows from the double and the triple formulas and level calculations by the standard group theoretic arguments. Not to repeat these routine arguments in future, in §10 we do this part of the proof axiomatically, for all group functors enjoying some formal properties. Finally, in

§11 we indicate some further possible applications of our results and state some unsolved problems.

## 2. FORM RINGS AND FORM IDEAL

The notion of  $\Lambda$ -quadratic forms, quadratic modules and generalised unitary groups over a form ring  $(A, \Lambda)$  were introduced by Anthony Bak in his Thesis, see [1, 2]. In this section, and the next one, we *very briefly* review the most fundamental notation and results that will be constantly used in the present paper. We refer to [2, 19, 33, 9, 20, 21, 28, 53, 35] for details, proofs, and further references.

2.1. Let  $R$  be a commutative ring with 1, and  $A$  be an (not necessarily commutative)  $R$ -algebra. An involution, denoted by  $\bar{\phantom{x}}$ , is an anti-homomorphism of  $A$  of order 2. Namely, for  $\alpha, \beta \in A$ , one has  $\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$ ,  $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$  and  $\bar{\bar{\alpha}} = \alpha$ . Fix an element  $\lambda \in \text{Cent}(A)$  such that  $\lambda\bar{\lambda} = 1$ . One may define two additive subgroups of  $A$  as follows:

$$\Lambda_{\min} = \{\alpha - \lambda\bar{\alpha} \mid \alpha \in A\}, \quad \Lambda_{\max} = \{\alpha \in A \mid \alpha = -\lambda\bar{\alpha}\}.$$

A *form parameter*  $\Lambda$  is an additive subgroup of  $A$  such that

- (1)  $\Lambda_{\min} \subseteq \Lambda \subseteq \Lambda_{\max}$ ,
- (2)  $\alpha\Lambda\bar{\alpha} \subseteq \Lambda$  for all  $\alpha \in A$ .

The pair  $(A, \Lambda)$  is called a *form ring*.

2.2. Let  $I \trianglelefteq A$  be a two-sided ideal of  $A$ . We assume  $I$  to be involution invariant, i. e. such that  $\bar{I} = I$ . Set

$$\Gamma_{\max}(I) = I \cap \Lambda, \quad \Gamma_{\min}(I) = \{\xi - \lambda\bar{\xi} \mid \xi \in I\} + \langle \xi\alpha\bar{\xi} \mid \xi \in I, \alpha \in \Lambda \rangle.$$

A *relative form parameter*  $\Gamma$  in  $(A, \Lambda)$  of level  $I$  is an additive group of  $I$  such that

- (1)  $\Gamma_{\min}(I) \subseteq \Gamma \subseteq \Gamma_{\max}(I)$ ,
- (2)  $\alpha\Gamma\bar{\alpha} \subseteq \Gamma$  for all  $\alpha \in A$ .

The pair  $(I, \Gamma)$  is called a *form ideal*.

In the level calculations we will use sums and products of form ideals. Let  $(I, \Gamma)$  and  $(J, \Delta)$  be two form ideals. Their sum is artlessly defined as  $(I + J, \Gamma + \Delta)$ , it is immediate to verify that this is indeed a form ideal.

Guided by analogy, one is tempted to set  $(I, \Gamma)(J, \Delta) = (IJ, \Gamma\Delta)$ . However, it is considerably harder to correctly define the product of two relative form parameters. The papers [17, 18, 20, 21] introduce the following definition

$$\Gamma\Delta = \Gamma_{\min}(IJ) + {}^J\Gamma + {}^I\Delta,$$

where

$${}^J\Gamma = \langle \xi\Gamma\bar{\xi} \mid \xi \in J \rangle, \quad {}^I\Delta = \langle \xi\Delta\bar{\xi} \mid \xi \in I \rangle.$$

One can verify that this is indeed a relative form parameter of level  $IJ$  if  $IJ = JI$ .

However, in the present paper we do not wish to impose any such commutativity assumptions. Thus, we are forced to consider the symmetrised products

$$I \circ J = IJ + JI, \quad \Gamma \circ \Delta = \Gamma_{\min}(IJ + JI) + {}^J\Gamma + {}^I\Delta$$

The notation  $\Gamma \circ \Delta$  – as also  $\Gamma\Delta$  is slightly misleading, since in fact it depends on  $I$  and  $J$ , not just on  $\Gamma$  and  $\Delta$ . Thus, strictly speaking, one should speak of the symmetrised products of *form ideals*

$$(I, \Gamma) \circ (J, \Delta) = (IJ + JI, \Gamma_{\min}(IJ + JI) + {}^J\Gamma + {}^I\Delta).$$

Clearly, in the above notation one has

$$(I, \Gamma) \circ (J, \Delta) = (I, \Gamma)(J, \Delta) + (J, \Delta)(I, \Gamma).$$

2.3. A *form algebra over a commutative ring  $R$*  is a form ring  $(A, \Lambda)$ , where  $A$  is an  $R$ -algebra and the involution leaves  $R$  invariant, i.e.,  $\overline{R} = R$ .

- A form algebra  $(A, \Lambda)$  is called *module finite*, if  $A$  is finitely generated as an  $R$ -module.
- A form algebra  $(A, \Lambda)$  is called *quasi-finite*, if there is a direct system of module finite  $R$ -subalgebras  $A_i$  of  $A$  such that  $\varinjlim A_i = A$ .

However, in general  $\Lambda$  is not an  $R$ -module. This forces us to replace  $R$  by its subring  $R_0$ , generated by all  $\alpha\overline{\alpha}$  with  $\alpha \in R$ . Clearly, all elements in  $R_0$  are invariant with respect to the involution, i. e.  $\overline{r} = r$ , for  $r \in R_0$ .

It is immediate, that any form parameter  $\Lambda$  is an  $R_0$ -module. This simple fact will be used throughout. This is precisely why we have to localise in multiplicative subsets of  $R_0$ , rather than in those of  $R$  itself.

2.4. Let  $(A, \Lambda)$  be a form algebra over a commutative ring  $R$  with 1, and let  $S$  be a multiplicative subset of  $R_0$ , (see §2.3). For any  $R_0$ -module  $M$  one can consider its localisation  $S^{-1}M$  and the corresponding localisation homomorphism  $F_S : M \rightarrow S^{-1}M$ . By definition of the ring  $R_0$  both  $A$  and  $\Lambda$  are  $R_0$ -modules, and thus can be localised in  $S$ .

In the present paper, we mostly use localisation with respect to the following two types of multiplication systems of  $R_0$ .

- *Principal localisation*: for any  $s \in R_0$  with  $\overline{s} = s$ , the multiplicative system generated by  $s$  is defined as  $\langle s \rangle = \{1, s, s^2, \dots\}$ . The localisation of the form algebra  $(A, \Lambda)$  with respect to multiplicative system  $\langle s \rangle$  is usually denoted by  $(A_s, \Lambda_s)$ , where as usual  $A_s = \langle s \rangle^{-1}A$  and  $\Lambda_s = \langle s \rangle^{-1}\Lambda$  are the usual principal localisations of the ring  $A$  and the form parameter  $\Lambda$ . Notice that, for each  $\alpha \in A_s$ , there exists an integer  $n$  and an element  $a \in A$  such that  $\alpha = \frac{a}{s^n}$ , and for each  $\xi \in \Lambda_s$ , there exists an integer  $m$  and an element  $\zeta \in \Lambda$  such that  $\xi = \frac{\zeta}{s^m}$ .

- *Maximal localisation*: consider a maximal ideal  $\mathfrak{m} \in \text{Max}(R_0)$  of  $R_0$  and the multiplicative closed set  $S_{\mathfrak{m}} = R_0 \setminus \mathfrak{m}$ . We denote the localisation of the form algebra  $(A, \Lambda)$  with respect to  $S_{\mathfrak{m}}$  by  $(A_{\mathfrak{m}}, \Lambda_{\mathfrak{m}})$ , where  $A_{\mathfrak{m}} = S_{\mathfrak{m}}^{-1}A$  and  $\Lambda_{\mathfrak{m}} = S_{\mathfrak{m}}^{-1}\Lambda$  are the usual maximal localisations of the ring  $A$  and the form parameter, respectively.

In these cases the corresponding localisation homomorphisms will be denoted by  $F_s$  and by  $F_{\mathfrak{m}}$ , respectively.

The following fact is verified by a straightforward computation.

**Lemma 1.** *For any  $s \in R_0$  and for any  $\mathfrak{m} \in \text{Max}(R_0)$  the pairs  $(A_s, \Lambda_s)$  and  $(A_{\mathfrak{m}}, \Lambda_{\mathfrak{m}})$  are form rings.*

### 3. UNITARY GROUPS

In the present section we recall basic notation and facts related to Bak's generalised unitary groups and their elementary subgroups.

3.1. Let, as above,  $A$  be an associative ring with 1. For natural  $m, n$  we denote by  $M(m, n, A)$  the additive group of  $m \times n$  matrices with entries in  $A$ . In particular  $M(m, A) = M(m, m, A)$  is the ring of matrices of degree  $m$  over  $A$ . For a matrix  $x \in M(m, n, A)$  we denote by  $x_{ij}$ ,  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , its entry in the position  $(i, j)$ . Let  $e$  be the identity matrix and  $e_{ij}$ ,  $1 \leq i, j \leq m$ , be a standard matrix unit, i.e. the matrix which has 1 in the position  $(i, j)$  and zeros elsewhere.

As usual,  $\mathrm{GL}(m, A) = M(m, A)^*$  denotes the general linear group of degree  $m$  over  $A$ . The group  $\mathrm{GL}(m, A)$  acts on the free right  $A$ -module  $V \cong A^m$  of rank  $m$ . Fix a base  $e_1, \dots, e_m$  of the module  $V$ . We may think of elements  $v \in V$  as columns with components in  $A$ . In particular,  $e_i$  is the column whose  $i$ -th coordinate is 1, while all other coordinates are zeros.

Actually, in the present paper we are only interested in the case, when  $m = 2n$  is even. We usually number the base as follows:  $e_1, \dots, e_n, e_{-n}, \dots, e_{-1}$ . All other occurring geometric objects will be numbered accordingly. Thus, we write  $v = (v_1, \dots, v_n, v_{-n}, \dots, v_{-1})^t$ , where  $v_i \in A$ , for vectors in  $V \cong A^{2n}$ .

The set of indices will be always ordered accordingly,  $\Omega = \{1, \dots, n, -n, \dots, -1\}$ . Clearly,  $\Omega = \Omega^+ \sqcup \Omega^-$ , where  $\Omega^+ = \{1, \dots, n\}$  and  $\Omega^- = \{-n, \dots, -1\}$ . For an element  $i \in \Omega$  we denote by  $\varepsilon(i)$  the sign of  $\Omega$ , i.e.  $\varepsilon(i) = +1$  if  $i \in \Omega^+$ , and  $\varepsilon(i) = -1$  if  $i \in \Omega^-$ .

3.2. For a form ring  $(A, \Lambda)$ , one considers the *hyperbolic unitary group*  $\mathrm{GU}(2n, A, \Lambda)$ , see [9, §2]. This group is defined as follows:

One fixes a symmetry  $\lambda \in \mathrm{Cent}(A)$ ,  $\lambda\bar{\lambda} = 1$  and supplies the module  $V = A^{2n}$  with the following  $\lambda$ -hermitian form  $h : V \times V \longrightarrow A$ ,

$$h(u, v) = \bar{u}_1 v_{-1} + \dots + \bar{u}_n v_{-n} + \lambda \bar{u}_{-n} v_n + \dots + \lambda \bar{u}_{-1} v_1.$$

and the following  $\Lambda$ -quadratic form  $q : V \longrightarrow A/\Lambda$ ,

$$q(u) = \bar{u}_1 u_{-1} + \dots + \bar{u}_n u_{-n} \pmod{\Lambda}.$$

In fact, both forms are engendered by a sesquilinear form  $f$ ,

$$f(u, v) = \bar{u}_1 v_{-1} + \dots + \bar{u}_n v_{-n}.$$

Now,  $h = f + \lambda \bar{f}$ , where  $\bar{f}(u, v) = \overline{f(v, u)}$ , and  $q(v) = f(u, u) \pmod{\Lambda}$ .

By definition, the hyperbolic unitary group  $\mathrm{GU}(2n, A, \Lambda)$  consists of all elements from  $\mathrm{GL}(V) \cong \mathrm{GL}(2n, A)$  preserving the  $\lambda$ -hermitian form  $h$  and the  $\Lambda$ -quadratic form  $q$ . In other words,  $g \in \mathrm{GL}(2n, A)$  belongs to  $\mathrm{GU}(2n, A, \Lambda)$  if and only if

$$h(gu, gv) = h(u, v) \quad \text{and} \quad q(gu) = q(u), \quad \text{for all } u, v \in V.$$

When the form parameter is not maximal or minimal, these groups are not algebraic. However, their internal structure is very similar to that of the usual classical groups. They are also oftentimes called general quadratic groups, or classical-like groups.

3.3. *Elementary unitary transvections*  $T_{ij}(\xi)$  correspond to the pairs  $i, j \in \Omega$  such that  $i \neq j$ . They come in two stocks. Namely, if, moreover,  $i \neq -j$ , then for any  $\xi \in A$  we set

$$T_{ij}(\xi) = e + \xi e_{ij} - \lambda^{(\varepsilon(j)-\varepsilon(i))/2} \bar{\xi} e_{-j, -i}.$$

These elements are also often called *elementary short root unipotents*. On the other side for  $j = -i$  and  $\alpha \in \lambda^{-(\varepsilon(i)+1)/2} \Lambda$  we set

$$T_{i, -i}(\alpha) = e + \alpha e_{i, -i}.$$

These elements are also often called *elementary long root elements*.

Note that  $\bar{\Lambda} = \bar{\lambda}\Lambda$ . In fact, for any element  $\alpha \in \Lambda$  one has  $\bar{\alpha} = -\bar{\lambda}\alpha$  and thus  $\bar{\Lambda}$  coincides with the set of products  $\bar{\lambda}\alpha$ ,  $\alpha \in \Lambda$ . This means that in the above definition  $\alpha \in \bar{\Lambda}$  when  $i \in \Omega^+$  and  $\alpha \in \Lambda$  when  $i \in \Omega^-$ .

Subgroups  $X_{ij} = \{T_{ij}(\xi) \mid \xi \in A\}$ , where  $i \neq \pm j$ , are called *short root subgroups*. Clearly,  $X_{ij} = X_{-j, -i}$ . Similarly, subgroups  $X_{i, -i} = \{T_{i, -i}(\alpha) \mid \alpha \in \lambda^{-(\varepsilon(i)+1)/2} \Lambda\}$  are called *long root subgroups*.

The *elementary unitary group*  $\text{EU}(2n, A, \Lambda)$  is generated by elementary unitary transvections  $T_{ij}(\xi)$ ,  $i \neq \pm j$ ,  $\xi \in A$ , and  $T_{i, -i}(\alpha)$ ,  $\alpha \in \Lambda$ , see [9, §3].

3.4. Elementary unitary transvections  $T_{ij}(\xi)$  satisfy the following *elementary relations*, also known as *Steinberg relations*. These relations will be used throughout this paper.

- (R1)  $T_{ij}(\xi) = T_{-j, -i}(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2} \bar{\xi})$ ,
- (R2)  $T_{ij}(\xi)T_{ij}(\zeta) = T_{ij}(\xi + \zeta)$ ,
- (R3)  $[T_{ij}(\xi), T_{hk}(\zeta)] = e$ , where  $h \neq j, -i$  and  $k \neq i, -j$ ,
- (R4)  $[T_{ij}(\xi), T_{jh}(\zeta)] = T_{ih}(\xi\zeta)$ , where  $i, h \neq \pm j$  and  $i \neq \pm h$ ,
- (R5)  $[T_{ij}(\xi), T_{j, -i}(\zeta)] = T_{i, -i}(\xi\zeta - \lambda^{-\varepsilon(i)} \bar{\zeta}\xi)$ , where  $i \neq \pm j$ ,
- (R6)  $[T_{i, -i}(\alpha), T_{-i, j}(\xi)] = T_{ij}(\alpha\xi)T_{-j, j}(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2} \bar{\xi}\alpha\xi)$ , where  $i \neq \pm j$ .

Relation (R1) coordinates two natural parametrisations of the same short root subgroup  $X_{ij} = X_{-j, -i}$ . Relation (R2) expresses additivity of the natural parametrisations. All other relations are various instances of the Chevalley commutator formula. Namely, (R3) corresponds to the case, where the sum of two roots is not a root, whereas (R4), and (R5) correspond to the case of two short roots, whose sum is a short root, and a long root, respectively. Finally, (R6) is the Chevalley commutator formula for the case of a long root and a short root, whose sum is a root. Observe that any two long roots are either opposite, or orthogonal, so that their sum is never a root.

3.5. Let  $G$  be a group. For any  $x, y \in G$ ,  ${}^x y = xyx^{-1}$  and  $y^x = x^{-1}yx$  denote the left conjugate and the right conjugate of  $y$  by  $x$ , respectively. As usual,  $[x, y] = xyx^{-1}y^{-1}$  denotes the left-normed commutator of  $x$  and  $y$ . Throughout the present paper we repeatedly use the following commutator identities:

$$(C1) \quad [x, yz] = [x, y] \cdot {}^y [x, z],$$

(C1<sup>+</sup>) An easy induction, using identity (C1), shows that

$$\left[ x, \prod_{i=1}^k u_i \right] = \prod_{i=1}^k \prod_{j=1}^{i-1} u_j [x, u_i],$$

where by convention  $\prod_{j=1}^0 u_j = 1$ ,

(C2)  $[xy, z] = {}^x[y, z] \cdot [x, z]$ ,

(C2<sup>+</sup>) As in (C1<sup>+</sup>), we have

$$\left[ \prod_{i=1}^k u_i, x \right] = \prod_{i=1}^k \prod_{j=1}^{k-i} u_j [u_{k-i+1}, x],$$

(C3)  ${}^x[[x^{-1}, y], z] \cdot {}^z[[z^{-1}, x], y] \cdot {}^y[[y^{-1}, z], x] = 1$ ,

(C4)  $[x, {}^y z] = {}^y[y^{-1}x, z]$ ,

(C5)  $[{}^y x, z] = {}^y[x, {}^{y^{-1}}z]$ ,

(C6) If  $H$  and  $K$  are subgroups of  $G$ , then  $[H, K] = [K, H]$ ,

Especially important is (C3), the celebrated *Hall–Witt identity*. Sometimes it is used in the following form, known as the *three subgroup lemma*.

**Lemma 2.** *Let  $F, H, L \trianglelefteq G$  be three normal subgroups of  $G$ . Then*

$$[[F, H], L] \leq [[F, L], H] \cdot [F, [H, L]].$$

#### 4. RELATIVE SUBGROUPS

In this section we recall definitions and basic facts concerning relative subgroups.

4.1. One associates with a form ideal  $(I, \Gamma)$  the following four relative subgroups.

- The subgroup  $\text{FU}(2n, I, \Gamma)$  generated by elementary unitary transvections of level  $(I, \Gamma)$ ,

$$\text{FU}(2n, I, \Gamma) = \langle T_{ij}(\xi) \mid \xi \in I \text{ if } i \neq \pm j \text{ and } \xi \in \lambda^{-(\varepsilon(i)+1)/2} \Gamma \text{ if } i = -j \rangle.$$

- The *relative elementary subgroup*  $\text{EU}(2n, I, \Gamma)$  of level  $(I, \Gamma)$ , defined as the normal closure of  $\text{FU}(2n, I, \Gamma)$  in  $\text{EU}(2n, A, \Lambda)$ ,

$$\text{EU}(2n, I, \Gamma) = \text{FU}(2n, I, \Gamma)^{\text{EU}(2n, A, \Lambda)}.$$

- The *principal congruence subgroup*  $\text{GU}(2n, I, \Gamma)$  of level  $(I, \Gamma)$  in  $\text{GU}(2n, A, \Lambda)$  consists of those  $g \in \text{GU}(2n, A, \Lambda)$ , which are congruent to  $e$  modulo  $I$  and preserve  $f(u, u)$  modulo  $\Gamma$ ,

$$f(gu, gu) \in f(u, u) + \Gamma, \quad u \in V.$$

- The full congruence subgroup  $\text{CU}(2n, I, \Gamma)$  of level  $(I, \Gamma)$ , defined as

$$\text{CU}(2n, I, \Gamma) = \{g \in \text{GU}(2n, A, \Lambda) \mid [g, \text{GU}(2n, A, \Lambda)] \subseteq \text{GU}(2n, I, \Gamma)\}.$$

In some books, including [19], the group  $\text{CU}(2n, I, \Gamma)$  is defined differently. However, in many important situations these definitions yield the same group.

4.2. Let us collect several basic facts, concerning relative groups, which will be used in the sequel. The first one of them asserts that the relative elementary groups are  $\text{EU}(2n, A, \Lambda)$ -perfect.

**Lemma 3.** *Suppose either  $n \geq 3$  or  $n = 2$  and  $I = \Lambda I + I\Lambda$ . Then*

$$\text{EU}(2n, I, \Gamma) = [\text{EU}(2n, I, \Gamma), \text{EU}(2n, A, \Lambda)].$$

The next lemma gives generators of the relative elementary subgroup  $\text{EU}(2n, I, \Gamma)$  as a subgroup. With this end, consider matrices

$$Z_{ij}(\xi, \zeta) = T_{ji}(\zeta)T_{ij}(\xi) = T_{ji}(\zeta)T_{ij}(\xi)T_{ji}(-\zeta),$$

where  $\xi \in I$ ,  $\zeta \in A$ , if  $i \neq \pm j$ , and  $\xi \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma$ ,  $\zeta \in \lambda^{-(\varepsilon(j)+1)/2}\Lambda$ , if  $i = -j$ . The following result is [9], Proposition 5.1.

**Lemma 4.** *Suppose  $n \geq 3$ . Then*

$$\begin{aligned} \text{EU}(2n, I, \Gamma) = \langle Z_{ij}(\xi, \zeta) \mid & \xi \in I, \zeta \in A \text{ if } i \neq \pm j \text{ and} \\ & \xi \in \lambda^{-(\varepsilon(i)+1)/2}\Gamma, \zeta \in \lambda^{-(\varepsilon(j)+1)/2}\Lambda, \text{ if } i = -j \rangle. \end{aligned}$$

The following lemma was first established in [1], but remained unpublished. See [19] and [9], Lemma 4.4, for published proofs.

**Lemma 5.** *The groups  $\text{GU}(2n, I, \Gamma)$  and  $\text{CU}(2n, I, \Gamma)$  are normal in  $\text{GU}(2n, A, \Lambda)$ .*

Also, throughout the paper we use the absolute and the relative standard commutator formulas, which were already stated in the introduction as Theorem 2 and Theorem 4.

4.3. The proofs in the present paper critically depend on the fact that the functors  $\text{GU}_{2n}$  and  $\text{EU}_{2n}$  commute with direct limits. This idea is used twice.

- Analysis of the quasi-finite case can be reduced to the case, where  $A$  is module finite over  $R_0$ , whereas  $R_0$  itself is Noetherian. Indeed, if  $(A, \Lambda)$  is quasi-finite, (see §2.3), it is a direct limit  $\varinjlim ((A_j)_{R_j}, \Lambda_j)$  of an inductive system of form sub-algebras  $((A_j)_{R_j}, \Lambda_j) \subseteq (A_R, \Lambda)$  such that each  $A_j$  is module finite over  $R_j$ ,  $R_0 \subseteq R_j$  and  $R_j$  is finitely generated as an  $R_0$ -module. It follows that  $A_j$  is finitely generated as an  $R_0$ -module, see [20, Cor. 3.8]. This reduction to module finite algebras will be used in Lemma 13 and Theorem 7.

- Analysis of any localisation can be reduced to the case of principal localisations. Indeed, let  $S$  be a multiplicative system in a commutative ring  $R$ . Then  $R_s$ ,  $s \in S$ , is an inductive system with respect to the localisation maps  $F_t : R_s \rightarrow R_{st}$ . Thus, for any functor  $\mathcal{F}$  commuting with direct limits one has  $\mathcal{F}(S^{-1}R) = \varinjlim \mathcal{F}(R_s)$ .

The following crucial lemma relies on both of these reductions. In fact, starting from the next section, we will be mostly working in the principal localisation  $A_t$ . However, eventually we shall have to return to the algebra  $A$  itself. In general, localisation homomorphism  $F_S$  is not injective, so we cannot pull elements of  $\text{GU}(2n, S^{-1}A, S^{-1}\Lambda)$  back to  $\text{GU}(2n, A, \Lambda)$ . However, over a Noetherian ring, principal localisation homomorphisms  $F_t$  are indeed injective on small  $t$ -adic neighbourhoods of identity!

**Lemma 6.** *Let  $R$  be a commutative Noetherian ring and let  $A$  be a module finite  $R$ -algebra. Then for any  $t \in R$  there exists a positive integer  $l$  such that restriction*

$$F_t : \text{GU}(2n, t^l A, t^l \Lambda) \rightarrow \text{GU}(2n, A_t, \Lambda_t),$$

of the localisation map to the principal congruence subgroup of level  $(t^l A, t^l \Lambda)$  is injective.

*Proof.* The proof follows from the injectivity of the localisation map  $F_t : t^l A \rightarrow A_t$ , see [3, Lemma 4.10].  $\square$

## 5. LEVELS OF MIXED COMMUTATOR SUBGROUPS

In the present section we closely follow the notation and computations of [29]. For the proof of Theorem 7 it is absolutely vital to improve the level calculations from [29], §8. Specifically, here we amalgamate Lemmas 22 and 23 therefrom, and streamline their proofs. As before, we assume that  $(A, \Lambda)$  is a form ring over a commutative ring  $R$  with involution,  $R_0$  is the subring of  $R$ , generated by  $a\bar{a}$ , where  $a \in R$ , as in §2.3,  $(I, \Gamma)$  and  $(J, \Delta)$  are two form ideals of  $(A, \Lambda)$  and, finally,  $n \geq 3$ . In this setting, in §2 we have defined the symmetrised product of form parameters  $\Gamma$  and  $\Delta$  as

$$\Gamma \circ \Delta = {}^J\Gamma + {}^I\Delta + \Gamma_{\min}(IJ + JI),$$

which is a relative form parameter of level  $I \circ J = IJ + JI$ . Notice that for any form ideal  $(I, \Gamma)$  of the form ring  $(A, \Lambda)$ , we have  $\Gamma = \Gamma \circ \Lambda$ .

First, we recall the rough level calculation of mixed commutator subgroups, which was essentially contained already in [17, 18, 20, 21] and reproduced in more details in [29]. The left inclusion in the following lemma is [29], Lemma 21, while the right inclusion is [29], Lemma 23.

**Lemma 7.** *Let  $(I, \Gamma)$  and  $(J, \Delta)$  be two form ideals of a form ring  $(A, \Lambda)$ . Then*

$$\begin{aligned} \text{EU}(2n, (I, \Gamma) \circ (J, \Delta)) &\subseteq [\text{FU}(2n, I, \Gamma), \text{FU}(2n, J, \Delta)] \\ &\subseteq [\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)] \subseteq \text{GU}(2n, (I, \Gamma) \circ (J, \Delta)). \end{aligned}$$

Actually, [29], Lemma 23, asserted a bit more, namely that one occurrence of the relative elementary subgroup can be replaced by the corresponding principal congruence subgroup,

$$[\text{EU}(2n, I, \Gamma), \text{GU}(2n, J, \Delta)] \subseteq \text{GU}(2n, (I, \Gamma) \circ (J, \Delta)).$$

Does this inclusion hold when *both* relative elementary subgroup are replaced by the corresponding principal congruence subgroups? For all  $n \geq 2$  Lemma 22 of [29], established a similar, but weaker inclusion

$$[\text{GU}(2n, I, \Gamma), \text{GU}(2n, J, \Delta)] \subseteq \text{GU}(2n, I \circ J, \Gamma_{\max}(I \circ J)),$$

with the *maximal* relative form parameter of level  $I \circ J$  on the right hand side, instead of the symmetrised product of the relative form parameters. For  $n \geq 3$  we can in fact merge these results, but the argument is not straightforward, this is why we missed it when writing [29]. This argument refers to the structure theorems for the *stable* unitary groups established in [1, 12], see also [64, 28].

**Lemma 8.** *Let  $(A, \Lambda)$  be a form ring and  $(I, \Gamma)$  and  $(J, \Delta)$  be form ideals of  $(A, \Lambda)$ . Then we have*

$$[\text{GU}(2n, I, \Gamma), \text{GU}(2n, J, \Delta)] \subseteq \text{GU}(2n, (I, \Gamma) \circ (J, \Delta)). \quad (6)$$

*Proof.* We first show that (6) holds for the *stable* unitary groups, namely that

$$[\mathrm{GU}(I, \Gamma), \mathrm{GU}(J, \Delta)] \subseteq \mathrm{GU}((I, \Gamma) \circ (J, \Delta)). \quad (7)$$

By the stable analogue of Lemma 7, which immediately follows by passage to limits, we have inclusions

$$\mathrm{EU}((I, \Gamma) \circ (J, \Delta)) \subseteq [\mathrm{EU}(I, \Gamma), \mathrm{EU}(J, \Delta)] \subseteq [\mathrm{GU}(I, \Gamma), \mathrm{GU}(J, \Delta)] \quad (8)$$

and

$$[\mathrm{EU}(I, \Gamma), \mathrm{EU}(J, \Delta)] \subseteq \mathrm{GU}((I, \Gamma) \circ (J, \Delta)). \quad (9)$$

Since the subgroup  $[\mathrm{GU}(I, \Gamma), \mathrm{GU}(J, \Delta)]$  is normalized by  $E(A, \Lambda)$ , applying Bass' sandwich theorem, see [19, Theorem 5.4.10], we can conclude that there exists a unique form ideal  $(K, \Omega)$  such that

$$\mathrm{EU}(K, \Omega) \subseteq [\mathrm{GU}(I, \Gamma), \mathrm{GU}(J, \Delta)] \subseteq \mathrm{GU}(K, \Omega). \quad (10)$$

By Lemma 2, we get

$$\begin{aligned} & [[\mathrm{GU}(I, \Gamma), \mathrm{GU}(J, \Delta)], \mathrm{EU}(A, \Lambda)] \subseteq \\ & \quad [[\mathrm{GU}(I, \Gamma), \mathrm{EU}(A, \Lambda)], \mathrm{GU}(J, \Delta)] \cdot [[\mathrm{GU}(J, \Delta), \mathrm{EU}(A, \Lambda)], \mathrm{GU}(I, \Gamma)]. \end{aligned}$$

But the absolute commutator formula implies that

$$\begin{aligned} & [[\mathrm{GU}(I, \Gamma), \mathrm{EU}(A, \Lambda)], \mathrm{GU}(J, \Delta)] \cdot [[\mathrm{GU}(J, \Delta), \mathrm{EU}(A, \Lambda)], \mathrm{GU}(I, \Gamma)] = \\ & \quad [\mathrm{EU}(I, \Gamma), \mathrm{EU}(J, \Delta)]. \end{aligned} \quad (11)$$

Thus,

$$[[\mathrm{GU}(I, \Gamma), \mathrm{GU}(J, \Delta)], \mathrm{EU}(A, \Lambda)] \subseteq [\mathrm{EU}(I, \Gamma), \mathrm{EU}(J, \Delta)]. \quad (12)$$

Again by the general commutator formula and (9), we have

$$\begin{aligned} \mathrm{EU}((I, \Gamma) \circ (J, \Delta)) &= [\mathrm{EU}((I, \Gamma) \circ (J, \Delta)), \mathrm{EU}(A, \Lambda)] \\ &\subseteq [[\mathrm{EU}(I, \Gamma), \mathrm{EU}(J, \Delta)], \mathrm{EU}(A, \Lambda)] \\ &\subseteq [\mathrm{GU}((I, \Gamma) \circ (J, \Delta)), \mathrm{EU}(A, \Lambda)] = \mathrm{EU}((I, \Gamma) \circ (J, \Delta)). \end{aligned} \quad (13)$$

Forming another commutator of (12) with  $\mathrm{EU}(A, \Lambda)$  and applying the inequalities obtained in (13) we get

$$\left[ [[\mathrm{GU}(I, \Gamma), \mathrm{GU}(J, \Delta)], \mathrm{EU}(A, \Lambda)], \mathrm{EU}(A, \Lambda) \right] = \mathrm{EU}((I, \Gamma) \circ (J, \Delta)).$$

Using inclusions (10), we see that

$$\begin{aligned} \mathrm{EU}(K, \Omega) &= [[\mathrm{EU}(K, \Omega), \mathrm{EU}(A, \Lambda)], \mathrm{EU}(A, \Lambda)] \\ &\subseteq \left[ [[\mathrm{GU}(I, \Gamma), \mathrm{GU}(J, \Delta)], \mathrm{EU}(A, \Lambda)], \mathrm{EU}(A, \Lambda) \right] = \mathrm{EU}((I, \Gamma) \circ (J, \Delta)) \\ &= [[\mathrm{EU}((I, \Gamma) \circ (J, \Delta)), \mathrm{EU}(A, \Lambda)], \mathrm{EU}(A, \Lambda)] \\ &\subseteq \left[ [[\mathrm{GU}(I, \Gamma), \mathrm{GU}(J, \Delta)], \mathrm{EU}(A, \Lambda)], \mathrm{EU}(A, \Lambda) \right] \\ &\subseteq [[\mathrm{GU}(K, \Omega), \mathrm{EU}(A, \Lambda)], \mathrm{EU}(A, \Lambda)] = \mathrm{EU}(K, \Omega). \end{aligned}$$

Thus, we can conclude that  $\mathrm{EU}(K, \Omega) = \mathrm{EU}((I, \Gamma) \circ (J, \Delta))$ . This implies that  $(K, \Omega) = (I, \Gamma) \circ (J, \Delta)$ , see the second paragraph of the proof of [19, Theorem 5.4.10]. Substituting this equality in (10), we see that inclusion (7) holds at the stable level, as claimed.

Let  $\varphi$  denote the usual stability embedding  $\varphi : \mathrm{GU}(2n, A, \Lambda) \rightarrow \mathrm{GU}(A, \Lambda)$ . Then

$$\varphi([\mathrm{GU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)]) = [\varphi(\mathrm{GU}(2n, I, \Gamma)), \varphi(\mathrm{GU}(2n, J, \Delta))] \subset [\mathrm{GU}(I, \Gamma), \mathrm{GU}(J, \Delta)].$$

In particular, the result at the stable level implies that

$$\varphi([\mathrm{GU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)]) \subseteq \varphi(\mathrm{GU}(2n, A, \Lambda)) \cap \mathrm{GU}((I, \Gamma) \circ (J, \Delta)).$$

On the other hand,

$$\varphi(\mathrm{GU}(2n, A, \Lambda)) \cap \mathrm{GU}((I, \Gamma) \circ (J, \Delta)) = \varphi(\mathrm{GU}(2n, (I, \Gamma) \circ (J, \Delta))).$$

Since  $\varphi$  is injective, we can conclude that

$$[\mathrm{GU}(2n, I, \Gamma), \mathrm{GU}(2n, J, \Delta)] \subseteq \mathrm{GU}(2n, (I, \Gamma) \circ (J, \Delta)),$$

as claimed.  $\square$

## 6. GENERATION OF MIXED COMMUTATOR SUBGROUPS

Our next result is a higher analogue of Lemma 4. Despite its technical character, it is one of the main new tools of our proof. Actually, for applications to width problems it is more expedient to construct a shorter set of generators, and this will be done in our subsequent paper. The form below is especially adjusted for the version of the relative commutator calculus we cultivate in the two following sections.

**Theorem 9.** *Let  $(A, \Lambda)$  be a form ring and  $(I, \Gamma)$ ,  $(J, \Delta)$  be two form ideals of  $(A, \Lambda)$ . Then the mixed commutator subgroup  $[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)]$  is generated as a group by the elements of the form*

- ${}^c[T_{ji}(\alpha), {}^{T_{ij}(a)}T_{ji}(\beta)],$
- ${}^c[T_{ji}(\alpha), T_{ij}(\beta)],$
- ${}^cT_{ij}(\xi),$

where  $T_{ji}(\alpha) \in \mathrm{EU}(2n, I, \Gamma)$ ,  $T_{ji}(\beta) \in \mathrm{EU}(2n, J, \Delta)$ ,  $T_{ij}(\xi) \in \mathrm{EU}(2n, (I, \Gamma) \circ (J, \Delta))$ , and  $T_{ij}(a), c \in \mathrm{EU}(2n, A, \Lambda)$ .

*Proof.* A typical generator of  $[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)]$  is of the form  $[e, f]$ , where  $e \in \mathrm{EU}(2n, I, \Gamma)$  and  $f \in \mathrm{EU}(2n, J, \Delta)$ . Thanks to Lemma 4, we may assume that  $e$  and  $f$  are products of elements of the form

$$e_i = Z_{pq}(a, \alpha), \quad f_j = Z_{rs}(b, \beta),$$

where  $a \in A$  and  $\alpha \in I$  if  $p \neq \pm q$  and  $a \in \lambda^{-(\varepsilon(p)+1)/2}\Lambda$ ,  $\alpha \in \lambda^{-(\varepsilon(-p)+1)/2}\Gamma$  if  $p = -q$ , and where  $b \in A$  and  $\beta \in J$  if  $r \neq \pm s$  and  $b \in \lambda^{-(\varepsilon(r)+1)/2}\Lambda$ ,  $\beta \in \lambda^{-(\varepsilon(-r)+1)/2}\Delta$  if  $r = -s$ .

Applying (C1<sup>+</sup>) and then (C2<sup>+</sup>), one gets that  $[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)]$  is generated by the elements of the form

$${}^c[{}^{T_{pq}(a)}T_{qp}(\alpha), {}^{T_{ij}(b)}T_{ji}(\beta)],$$

where  $c \in \mathrm{EU}(2n, A, \Lambda)$ . Furthermore,

$${}^c[Z_{pq}(a, \alpha), {}^{T_{ij}(b)}T_{ji}(\beta)] = {}^{cT_{pq}(a)}[T_{qp}(\alpha), {}^{T_{pq}(-a)T_{ij}(b)}T_{ji}(\beta)].$$

The normality of  $\text{EU}(2n, J, \Delta)$  implies that  $T_{pq}(-a)T_{ij}(b)T_{ji}(\beta) \in \text{EU}(2n, J, \Delta)$ , which is a product of  $Z_{ij}(\xi, \zeta)$  by Lemma 4. Again by  $(C1^+)$ , one reduces the proof to the case of showing that

$$[T_{pq}(\alpha), Z_{ij}(b, \beta)]$$

is a product of the generators listed above. We divide the proof into 3 cases, namely,

- I.  $T_{pq}(\alpha)$  and  $Z_{ij}(b, \beta)$  have opposite indices, namely  $p = j$  and  $q = i$ .
- II.  $T_{pq}(\alpha)$  and  $Z_{ij}(b, \beta)$  have the same indices.
- III. Otherwise.

Case I,  $T_{pq}(\alpha)$  and  $Z_{ij}(b, \beta)$  are opposite, then  $p = j$  and  $q = i$ .

$$[T_{ji}(\alpha), Z_{ij}(b, \beta)] = [T_{ji}(\alpha), T_{ji}^{(b)}T_{ij}(\beta)] = T_{ji}^{(b)}[T_{ji}(\alpha), T_{ij}(\beta)],$$

which is a generator listed in the current theorem.

Case II, there is nothing to prove.

Case III, the proof can be further subdivided into the following subcases:

- 1.  $T_{pq}(\alpha)$  commutes with  $Z_{ij}(b, \beta)$ .  $[T_{pq}(\alpha), Z_{ij}(b, \beta)] = 1$  which satisfies the lemma.
- 2.  $T_{pq}(\alpha)$  and  $Z_{ij}(b, \beta)$  are short roots,  $q = i$  and  $p \neq \pm j$ . Using (C4) and (R3), one gets

$$\begin{aligned} [T_{pi}(\alpha), Z_{ij}(b, \beta)] &= [T_{pi}(\alpha), T_{ji}^{(b)}T_{ij}(\beta)] \\ &= T_{ji}^{(b)}[T_{pi}(\alpha)T_{ji}^{(b)}, T_{ij}(\beta)] \\ &= T_{ji}^{(b)}[T_{pi}(\alpha), T_{ij}(\beta)] \\ &= T_{ji}^{(b)}T_{pj}(\alpha\beta), \end{aligned}$$

which is a generator of  $[\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)]$ .

- 3.  $T_{pq}(\alpha)$  and  $Z_{ij}(b, \beta)$  are short roots,  $q = i$  and  $p = -j$ . Using (C4) and (R5), one gets

$$\begin{aligned} [T_{-j,i}(\alpha), Z_{ij}(b, \beta)] &= [T_{-j,i}(\alpha), T_{ji}^{(b)}T_{ij}(\beta)] \\ &= T_{ji}^{(b)}[T_{-j,i}(\alpha)T_{ji}^{(b)}, T_{ij}(\beta)] \\ &= T_{ji}^{(b)}[T_{-j,i}(\alpha), T_{ij}(\beta)] \\ &= T_{ji}^{(b)}T_{-j,j}(\alpha\beta - \lambda^{\varepsilon(j)}\overline{\beta\alpha}), \end{aligned}$$

which is of the form in the theorem.

- 4.  $T_{pq}(\alpha)$  and  $Z_{ij}(b, \beta)$  are short roots,  $p = j$  and  $q \neq \pm i$ . Using (R1), we reduce our consideration to the subcase (2).
- 5.  $T_{pq}(\alpha)$  and  $Z_{ij}(b, \beta)$  are short roots,  $p = j$  and  $q \neq -i$ . Using (R1), we reduce our consideration to the subcase (3).

6.  $T_{pq}(\alpha)$  is a long root and  $Z_{ij}(b, \beta)$  is a short root,  $q = i$ . Using (R6), we get

$$\begin{aligned}
[T_{-i,i}(\alpha), Z_{ij}(b, \beta)] &= [T_{-i,i}(\alpha), T_{ji}^{(b)} T_{ij}(\beta)] \\
&= T_{ji}^{(b)} [T_{-i,i}(\alpha) T_{ji}^{(b)}, T_{ij}(\beta)] \\
&= T_{ji}^{(b)} [T_{-i,i}(\alpha), T_{ij}(\beta)] \\
&= T_{ji}^{(b)} (T_{-i,j}(\alpha\beta) T_{-j,j}(-\lambda^{(\varepsilon(j)+\varepsilon(i))/2} \beta \alpha \bar{\beta})) \\
&= (T_{ji}^{(b)} T_{-i,j}(\alpha\beta)) (T_{ji}^{(b)} T_{-j,j}(-\lambda^{(\varepsilon(j)-\varepsilon(i))/2} \beta \alpha \bar{\beta})),
\end{aligned}$$

which is a product of the generators in the lemma.

7.  $T_{pq}(\alpha)$  is a long root and  $Z_{ij}(b, \beta)$  is a short root,  $p = j$ . Using (R1), we reduce our consideration to the subcase (6).  
 8.  $T_{pq}(\alpha)$  is a short root and  $Z_{ij}(b, \beta)$  is a long root,  $q = i$ . Using (R6), we have

$$\begin{aligned}
[T_{pi}(\alpha), Z_{i,-i}(b, \beta)] &= [T_{pi}(\alpha), T_{-i,i}^{(b)} T_{i,-i}(\beta)] \\
&= T_{-i,i}^{(b)} [T_{pi}(\alpha) T_{-i,i}^{(b)}, T_{i,-i}(\beta)] \\
&= T_{-i,i}^{(b)} [T_{pi}(\alpha), T_{i,-i}(\beta)] \\
&= T_{-i,i}^{(b)} (T_{p,-i}(\alpha\beta) T_{p,-p}(-\lambda^{(\varepsilon(i)-\varepsilon(p))/2} \alpha \beta \bar{\alpha})) \\
&= (T_{-i,i}^{(b)} T_{p,-i}(\alpha\beta)) (T_{-i,i}^{(b)} T_{p,-p}(-\lambda^{(\varepsilon(i)-\varepsilon(p))/2} \alpha \beta \bar{\alpha})),
\end{aligned}$$

which is a product of the generators in the lemma.

9.  $T_{pq}(\alpha)$  is a short root and  $Z_{ij}(b, \beta)$  is a long root,  $p = j$ . Using (R1), we reduce it to the subcase (8). This finishes the proof of Case III, hence the whole proof.  $\square$

## 7. COMMUTATOR CALCULUS

This and the next two sections constitute the technical heart of the paper.

Let us recall some facts from [29]. For any  $t \neq 0 \in R_0$  and any given positive integer  $l$ , the set  $t^l A$  is in fact an ideal of the algebra  $A$ . Similarly, it is straightforward to verify that  $t^l \Lambda = \{t^l \alpha \mid \alpha \in \Lambda\}$  is in fact a relative form parameter for  $t^l A$ , and, thus,  $(t^l A, t^l \Lambda)$  is a form ideal. This allows us to define the corresponding groups  $\text{FU}(2n, t^l A, t^l \Lambda)$  and  $\text{FU}(2n, t^l I, t^l \Gamma)$ . To make calculations somewhat less painful, we introduce the group  $\text{FU}(2n, t^l A, t^l I, t^l \Gamma)$  which by definition is the normal closure of  $\text{FU}(2n, t^l I, t^l \Gamma)$  in  $\text{FU}(2n, t^l A, t^l \Lambda)$ ,

$$\text{FU}(2n, t^l A, t^l I, t^l \Gamma) = {}^{\text{FU}(2n, t^l A, t^l \Lambda)} \text{FU}(2n, t^l I, t^l \Gamma) \trianglelefteq \text{FU}(2n, t^l A, t^l \Lambda).$$

Actually, the use of this base of  $t$ -adic neighbourhoods instead of the usual ones is precisely one of the key technical tricks of [31, 29, 30]. Normality of  $\text{FU}(2n, t^l A, t^l I, t^l \Gamma)$  in  $\text{FU}(2n, t^l A, t^l \Lambda)$  will be repeatedly used in the sequel. Observe, that  $\text{FU}(2n, t^l A, t^l A, t^l \Lambda) = \text{FU}(2n, t^l A, t^l \Lambda)$ .

Let us introduce a further piece of notation. For a form ideal  $(I, \Gamma)$  and an element  $t \in R_0$ , the set  $\text{FU}^1\left(2n, \frac{I}{t^m}, \frac{\Gamma}{t^m}\right)$  consists of elementary unitary transvections  $T_{ij}(a)$ , such that  $a \in \frac{I}{t^m}$  if  $i \neq \pm j$  and  $a \in \lambda^{\varepsilon(i)+1/2} \frac{\Gamma}{t^m}$  if  $i = -j$ . Denote by  $\text{FU}^L\left(2n, \frac{A}{t^m}, \frac{I}{t^m}, \frac{\Gamma}{t^m}\right)$  the set of products of  $\leq L$  elements of the form  $\text{FU}^1\left(2n, \frac{A}{t^m}, \frac{A}{t^m}\right) \text{FU}^1\left(2n, \frac{I}{t^m}, \frac{\Gamma}{t^m}\right)$ . The set

$\text{FU}^1(2n, t^m I, t^m \Gamma)$  is defined similarly. By the same token,  $\text{FU}^K(2n, t^m I, t^m \Gamma)$ , denotes the set of products of  $\leq K$  elements of  $\text{FU}^1(2n, t^m I, t^m \Gamma)$ .

The next result is a summary of the relative conjugation calculus and relative commutator calculus, as developed in [29], Lemmas 8, 11 and 12.

**Lemma 9.** *For any given  $m, l$  there exists a sufficiently large integer  $p$  such that*

$$\text{FU}^1\left(2n, \frac{A}{t^m}, \frac{\Lambda}{t^m}\right) \text{FU}(2n, t^p I, t^p \Gamma) \leq \text{FU}(2n, t^l A, t^l I, t^l \Gamma), \quad (14)$$

*there exists an integer  $p$  such that*

$$\begin{aligned} \text{FU}^1\left(2n, \frac{A}{t^m}, \frac{\Lambda}{t^m}\right) [\text{FU}(2n, t^p A, t^p I, t^p \Gamma), \text{FU}(2n, t^p A, t^p J, t^p \Delta)] \\ \subseteq [\text{FU}(2n, t^l A, t^l I, t^l \Gamma), \text{FU}(2n, t^l A, t^l J, t^l \Delta)], \end{aligned} \quad (15)$$

*and there exists an integer  $p$  such that*

$$\begin{aligned} [\text{FU}(2n, t^p A, t^p I, t^p \Gamma), \text{FU}^1\left(2n, \frac{J}{t^m}, \frac{\Delta}{t^m}\right)] \\ \subseteq [\text{FU}(2n, t^l A, t^l I, t^l \Gamma), \text{FU}(2n, t^l A, t^l J, t^l \Delta)]. \end{aligned} \quad (16)$$

With the use of the Hall–Witt identity (C3) this lemma immediately implies the following result.

**Lemma 10.** *For any given  $m, l, L$  there exists a sufficiently large integer  $p$  such that*

$$\begin{aligned} [\text{FU}(2n, t^p A, t^p I, t^p \Gamma), \text{FU}^L\left(2n, \frac{A}{t^m}, \frac{J}{t^m}, \frac{\Delta}{t^m}\right)] \\ \subseteq [\text{FU}(2n, t^l A, t^l I, t^l \Gamma), \text{FU}(2n, t^l A, t^l J, t^l \Delta)]. \end{aligned}$$

In the following three Lemmas, as in Lemma 9, all the calculations take place in the fraction ring  $(A_t, \Lambda_t)$  (see §2.4). All the subgroups of  $\text{GU}(2n, A_t, \Lambda_t)$  used in the Lemmas, such as  $\text{EU}(2n, A, I, \Gamma)$  or  $\text{GU}(2n, J, \Delta)$  are in fact the homomorphic images of the similar subgroups in  $\text{GU}(2n, A_t, \Lambda_t)$  under the natural homomorphism  $A \rightarrow A_t$ . Since lemmas such as Lemma 7 and the generalized commutator formula (Theorem 4) hold for these subgroups in  $\text{GU}(2n, A, \Lambda)$ , they also hold for their corresponding homomorphic images in  $\text{GU}(2n, A_t, \Lambda_t)$ .

**Lemma 11.** *Let  $(A, \Lambda)$  be a form algebra,  $(I, \Gamma)$  and  $(J, \Delta)$  form ideals of  $(A, \Lambda)$ ,  $m$  an integer and  $t \in R_0$ . If  $x \in \text{FU}^1(2n, \frac{I}{t^m}, \frac{\Gamma}{t^m})$ ,  $l$  is a given integer, then for every integer  $p \geq l + m$  we have*

$$[x, h] \in \text{GU}(2n, t^l(JI + IJ), t^l(\Gamma \circ \Delta)),$$

*where  $h \in \text{GU}(2n, t^p J, t^p \Delta)$ .*

*Proof.* Suppose that  $x = T_{sk}(\alpha)$ ,  $\alpha \in \frac{I}{t^m}$  for  $s \neq -k$ , and  $\alpha \in \lambda^{-(\varepsilon(s)+1)/2} \frac{\Gamma}{t^m}$  for  $s = -k$ . Let

$$g = [x, h].$$

By a similar argument as used in Lemma 13 in [32], it suffices to verify that there exists an integer  $p$ , such that

$$\sum_{1 \leq i \leq n} \bar{g}_{ij} g_{-i,j} \in t^l(\Gamma \circ \Delta)$$

for any given  $j$  with  $-n \leq j \leq n$ . We divide the proof into 2 cases according to the type of  $T_{sk}(\alpha)$ , namely long or short root. We provide a detailed calculation for the case of a long root type element. The case of a short root type element is settled by a similar calculation which will be omitted.

Case I. If  $T_{sk}(\alpha)$  is a long root, i.e.,  $s = -k$  and  $\alpha \in \lambda^{-(\varepsilon(s)+1)/2} \frac{\Gamma}{t^m}$ , then

$$g = [T_{s,-s}(\alpha), h] = T_{s,-s}(\alpha) \left( e - \sum_{i,j} h_{i,s} \alpha \bar{h}_{-j,s} \right),$$

where  $h_{ij} \in t^p I$ .

Let us have a closer look at the sum  $\sum_{1 \leq i \leq n} \bar{g}_{ij} g_{-i,j}$ . When  $j \neq -s$ , we may, without loss of generality, assume that  $s \geq 0$  and  $j \geq 0$ , and thus this sum can be rewritten in the form

$$\begin{aligned} & \sum_{1 \leq i \leq n} \overline{h_{i,s} \alpha \bar{h}_{-j,s} h_{-i,s} \alpha \bar{h}_{-j,s}} - \lambda^{(\varepsilon(j)-\varepsilon(-s))/2} h_{-j,s} \alpha \bar{h}_{-j,s} + \overline{\alpha h_{-s,s} \alpha \bar{h}_{-j,s} h_{-s,s} \alpha \bar{h}_{-j,s}} \\ &= \sum_{1 \leq i \leq n} h_{-j,s} \lambda \bar{\alpha} \bar{h}_{i,s} h_{-i,s} \alpha \bar{h}_{-j,s} - h_{-j,s} \bar{\lambda} \alpha \bar{h}_{-j,s} + h_{-j,s} \bar{\alpha} \bar{h}_{-s,s} \bar{\alpha} h_{-s,s} \alpha \bar{h}_{-j,s}, \end{aligned}$$

where the first summand belongs to  $t^{4p-2m}(I\Delta)$ , whereas the second and the third ones belong to  $t^{2p-m}(J\Gamma)$  and  $t^{4p-3m}(J\Gamma)$ , respectively.

On the other hand, when  $j = -s$ , this sum equals

$$\sum_{1 \leq i \leq n} \overline{h_{i,s} \alpha \bar{h}_{ss} h_{-i,s} \alpha \bar{h}_{ss}} - h_{ss} \bar{\alpha} \bar{h}_{ss} + (\bar{\alpha} - \overline{\alpha h_{-s,s} \alpha \bar{h}_{ss}})(1 - h_{-s,s} \alpha \bar{h}_{ss}),$$

where the first sum belongs to  $t^{4p-2m}(I\Delta)$ , while the rest equals

$$\begin{aligned} x &= -h_{ss} \bar{\alpha} \bar{h}_{ss} + (\bar{\alpha} - h_{ss} \bar{\alpha} \bar{h}_{-s,s} \bar{\alpha})(1 - h_{-s,s} \alpha \bar{h}_{ss}) = \\ &= -h_{ss} \bar{\alpha} \bar{h}_{ss} + \bar{\alpha} - \bar{\alpha} h_{-s,s} \alpha \bar{h}_{ss} - h_{ss} \bar{\alpha} \bar{h}_{-s,s} \bar{\alpha} + h_{ss} \bar{\alpha} \bar{h}_{-s,s} \bar{\alpha} h_{-s,s} \alpha \bar{h}_{ss} = \\ &= -(1 + h_{ss} - 1) \bar{\alpha} (1 + \overline{h_{ss} - 1}) + \bar{\alpha} + \left( \lambda \bar{\alpha} h_{-s,s} \bar{\alpha} \bar{h}_{ss} - h_{ss} \bar{\alpha} \bar{h}_{-s,s} \bar{\alpha} \right) + h_{ss} \bar{\alpha} \bar{h}_{-s,s} \bar{\alpha} h_{-s,s} \alpha \bar{h}_{ss} \end{aligned}$$

where the two last summands belong to  $t^{2p-2m}\Gamma_{\min}((IJ + JI))$  and to  $t^{4p-3m}(J\Gamma)$ , respectively. Thus, for the left summands, one has

$$\begin{aligned} & -(1 + h_{ss} - 1) \bar{\alpha} (1 + \overline{h_{ss} - 1}) + \bar{\alpha} \\ &= -(h_{ss} - 1) \bar{\alpha} + \lambda \alpha \overline{(h_{ss} - 1)} - (h_{ss} - 1) \alpha \overline{(h_{ss} - 1)}, \end{aligned}$$

where the first summand also belongs to  $t^{p-m}\Gamma_{\min}((IJ + JI))$ , whereas the second one belongs to  $t^{2p-m}(J\Gamma)$ , respectively.

Now by our assumption  $p \geq l + m$ , in both cases the desired sum belongs to  $t^l(\Gamma \circ \Delta)$ , as claimed.  $\square$

**Lemma 12.** *Let  $(A, \Lambda)$  be a module finite form algebra,  $(I, \Gamma)$ ,  $(J, \Delta)$  and  $(K, \Omega)$  form ideals of  $(A, \Lambda)$ , and  $t \in R_0$ . For any given integers  $m, l$ , there is a sufficiently large integer*

$p$ , such that

$$\begin{aligned} & \left[ [\text{FU}(2n, t^p A, t^p I, t^p \Gamma), \text{FU}(2n, t^p A, t^p J, t^p \Delta)], [\text{EU}(2n, A, \Lambda), \text{FU}^1(2n, \frac{K}{t^m}, \frac{\Omega}{t^m})] \right] \\ & \subseteq \left[ [\text{FU}(2n, t^l A, t^l I, t^l \Gamma), \text{FU}(2n, t^l A, t^l J, t^l \Delta)], \text{FU}(2n, t^l A, t^l K, t^l \Omega) \right]. \end{aligned}$$

*Proof.* Let  $x \in [\text{FU}(2n, t^p A, t^p I, t^p \Gamma), \text{FU}(2n, t^p A, t^p J, t^p \Delta)]$ ,  $y \in \text{EU}(2n, A, \Lambda)$  and  $z \in \text{FU}^1(2n, \frac{K}{t^m}, \frac{\Omega}{t^m})$ . Then using the Hall–Witt identity (C3), one obtain that

$$[x, [y^{-1}, z]] = y^{-1}x[[x^{-1}, y], z] \quad y^{-1}z[[z^{-1}, x], y]. \quad (17)$$

By our assumption and Lemma 7, we have

$$\begin{aligned} x & \in [\text{FU}(2n, t^p A, t^p I, t^p \Gamma), \text{FU}(2n, t^p A, t^p J, t^p \Delta)] \\ & \subseteq [\text{EU}(2n, t^p I, t^p \Gamma), \text{EU}(2n, t^p J, t^p \Delta)] \\ & \subseteq \text{GU}(2n, t^{2p}(I \circ J), t^{2p}(\Gamma \circ \Delta)). \end{aligned}$$

Using the commutator formula, we obtain that

$$\begin{aligned} [x^{-1}, y] & \in [\text{GU}(2n, t^{2p}(I \circ J), t^{2p}(\Gamma \circ \Delta)), \text{EU}(2n, A, \Lambda)] \\ & = \text{EU}(2n, t^{2p}(I \circ J), t^{2p}(\Gamma \circ \Delta)) \\ & = \text{EU}(2n, t^{2p}(I \circ J), \Lambda \circ t^{2p}(\Gamma \circ \Delta)). \end{aligned}$$

By Lemma 7, it follows that

$$\begin{aligned} & \text{EU}(2n, t^{2p}(I \circ J), \Lambda \circ t^{2p}(\Gamma \circ \Delta)) \\ & \subseteq [\text{FU}(2n, t^p A, t^p \Lambda), \text{FU}(2n, t^p(I \circ J), t^p(\Gamma \circ \Delta))] \\ & = \text{FU}(2n, t^p A, t^p(I \circ J), t^p(\Gamma \circ \Delta)). \end{aligned}$$

Therefore Lemma 10 implies that for any given  $p'$  there exists an integer  $p$  such that

$$\begin{aligned} [[x^{-1}, y], z] & \in y^{-1}x \left[ \text{FU}(2n, t^p A, t^p(I \circ J), t^p(\Gamma \circ \Delta)), \text{FU}^1(2n, \frac{K}{t^m}, \frac{\Omega}{t^m}) \right] \\ & \subseteq y^{-1}x \left[ \text{FU}(2n, t^{p'} A, t^{p'}(I \circ J), t^{p'}(\Gamma \circ \Delta)), \text{FU}(2n, t^{p'} A, t^{p'} K, t^{p'} \Omega) \right], \end{aligned}$$

where by definition,  $y^{-1}x \in \text{EU}(2n, \frac{A}{t^0}, \frac{\Lambda}{t^0})$ . By (15) in Lemma 9, for any given  $l$ , we may find a sufficiently large  $p'$  such that

$$\begin{aligned} & y^{-1}x \left[ \text{FU}(2n, t^{p'} A, t^{p'}(I \circ J), t^{p'}(\Gamma \circ \Delta)), \text{FU}(2n, t^{p'} A, t^{p'} K, t^{p'} \Omega) \right] \\ & \subseteq [\text{FU}(2n, t^{2l} A, t^{2l}(I \circ J), t^{2l}(\Gamma \circ \Delta)), \text{FU}(2n, t^{2l} A, t^{2l} K, t^{2l} \Omega)]. \end{aligned}$$

Now applying Lemma 7 again, we have

$$\begin{aligned} y^{-1}x[[x^{-1}, y], z] & \in [\text{FU}(2n, t^{2l} A, t^{2l}(I \circ J), t^{2l}(\Gamma \circ \Delta)), \text{FU}(2n, t^{2l} A, t^{2l} K, t^{2l} \Omega)] \\ & \subseteq [[\text{FU}(2n, t^l A, t^l I, t^l \Gamma), \text{FU}(2n, t^l A, t^l J, t^l \Delta)], \text{FU}(2n, t^{2l} A, t^{2l} K, t^{2l} \Omega)] \\ & \subseteq [[\text{FU}(2n, t^l A, t^l I, t^l \Gamma), \text{FU}(2n, t^l A, t^l J, t^l \Delta)], \text{FU}(2n, t^l A, t^l K, t^l \Omega)]. \end{aligned}$$

This proves the first factor of (17) satisfies the lemma.

For the second factor, using Lemma 7

$$\begin{aligned} [z^{-1}, x] &\in \left[ \text{FU}^1(2n, \frac{K}{t^m}, \frac{\Omega}{t^m}), [\text{FU}(2n, t^p A, t^p I, t^p \Gamma), \text{FU}(2n, t^p A, t^p J, t^p \Delta)] \right] \\ &\subseteq \left[ \text{FU}^1(2n, \frac{K}{t^m}, \frac{\Omega}{t^m}), \text{GU}(2n, t^{2p}(I \circ J), t^{2p}(\Gamma \circ \Delta)) \right]. \end{aligned}$$

Now applying Lemma 11, for any given integers  $l$  and  $m$ , there exists an integer  $p$  such that

$$\begin{aligned} &\left[ \text{FU}^1(2n, \frac{K}{t^m}, \frac{\Omega}{t^m}), \text{GU}(2n, t^p(I \circ J), t^p(\Gamma \circ \Delta)) \right] \\ &\subseteq \text{GU}(2n, t^{3p'}(K \circ (I \circ J)), t^{3p'}(\Omega \circ (\Gamma \circ \Delta))). \end{aligned}$$

Again by the commutator formula, we obtain

$$\begin{aligned} [[z^{-1}, x], y] &\in \left[ \text{GU}(2n, t^{3p'}(K \circ (I \circ J)), t^{3p'}(\Omega \circ (\Gamma \circ \Delta))), \text{EU}(2n, A, \Lambda) \right] \\ &= \text{EU}(2n, t^{3p'}(K \circ (I \circ J)), t^{3p'}(\Omega \circ (\Gamma \circ \Delta))). \end{aligned}$$

Applying Lemma 7 twice, we get

$$\begin{aligned} &\text{EU}(2n, t^{3p'}(K \circ (I \circ J)), t^{3p'}(\Omega \circ (\Gamma \circ \Delta))) \\ &\subseteq \left[ [\text{FU}(2n, t^{p'} I, t^{p'} \Gamma), \text{FU}(2n, t^{p'} J, t^{p'} \Delta)], \text{FU}(2n, t^{p'} K, t^{p'} \Omega) \right]. \end{aligned}$$

Finally, we have

$$\begin{aligned} y^{-1}z[[z^{-1}, x], y] &\in y^{-1}z \left[ [\text{FU}(2n, t^{p'} I, t^{p'} \Gamma), \text{FU}(2n, t^{p'} J, t^{p'} \Delta)], \text{FU}(2n, t^{p'} K, t^{p'} \Omega) \right] \\ &= \left[ [y^{-1}z \text{FU}(2n, t^{p'} I, t^{p'} \Gamma), y^{-1}z \text{FU}(2n, t^{p'} J, t^{p'} \Delta)], y^{-1}z \text{FU}(2n, t^{p'} K, t^{p'} \Omega) \right]. \end{aligned}$$

Now applying (15) in Lemma 9 to every component of the commutator above, we may find a sufficiently large  $p'$  such that for any given  $l$ ,

$$\begin{aligned} &\left[ [y^{-1}z \text{FU}(2n, t^{p'} I, t^{p'} \Gamma), y^{-1}z \text{FU}(2n, t^{p'} J, t^{p'} \Delta)], y^{-1}z \text{FU}(2n, t^{p'} K, t^{p'} \Omega) \right] \\ &\subseteq \left[ [\text{FU}(2n, t^l A, t^l I, t^l \Gamma), \text{FU}(2n, t^l A, t^l J, t^l \Delta)], \text{FU}(2n, t^l A, t^l K, t^l \Omega) \right]. \end{aligned}$$

This finishes the proof.  $\square$

## 8. MAIN LEMMA ON TRIPLE COMMUTATORS

The following lemma is crucial for proving the main result, i.e., Theorem 7 of this paper.

**Lemma 13.** *Let  $(A, \Lambda)$  be a module finite form algebra,  $(I, \Gamma)$ ,  $(J, \Delta)$  and  $(K, \Omega)$  form ideals of  $(A, \Lambda)$ , and  $t \in R_0$ . For any given  $e_2 \in \text{EU}(2n, K_t, \Omega_t)$  and integer  $l$ , there is a sufficiently large integer  $p$ , such that*

$$[e_1, e_2] \in \left[ [\text{EU}(2n, t^l I, t^l \Gamma), \text{EU}(2n, t^l J, t^l \Delta)], \text{EU}(2n, t^l K, t^l \Omega) \right], \quad (18)$$

where  $e_1 \in [\text{FU}^1(2n, t^p I, t^p \Gamma), \text{EU}(2n, J, \Delta)]$ .

*Proof.* For any given  $e_1 \in [\text{FU}^1(2n, t^p I, t^p \Gamma), \text{EU}(2n, J, \Delta)]$  and  $e_2 \in \text{EU}(2n, K_t, \Omega_t)$ , one may find some positive integers  $m, L$  and  $S$ , such that

$$e_1 \in [\text{FU}^1(2n, t^p I, t^p \Gamma), \text{FU}^S(2n, A, J, \Delta)]$$

and

$$e_2 \in \text{FU}^L(2n, \frac{A}{t^m}, \frac{K}{t^m}, \frac{\Omega}{t^m}).$$

Applying the identity (C1<sup>+</sup>) and repeated application of (14) in Lemma 9, we reduce the problem to show that

$$\begin{aligned} & \left[ [\text{FU}^1(2n, t^p I, t^p \Gamma), \text{FU}^S(2n, A, J, \Delta)], {}^c T_{i,j}(\frac{\gamma}{t^m}) \right] \subseteq \\ & \left[ [\text{EU}(2n, t^l I, t^l \Gamma), \text{EU}(2n, t^l J, t^l \Delta)], \text{EU}(2n, t^l K, t^l \Omega) \right], \end{aligned}$$

where  $c \in \text{FU}^1(2n, \frac{A}{t^m}, \frac{\Omega}{t^m})$  and  $T_{i,j}(\frac{\gamma}{t^m}) \in \text{EU}(2n, \frac{K}{t^m}, \frac{\Omega}{t^m})$ .

We claim that for any given integer  $p$ , there exists some integer  $m'$  such that any elementary root element  $T_{i,j}(\frac{\gamma}{t^m})$  can be further decomposed as a product

$$\left[ \text{FU}^1(2n, t^p A, t^p \Lambda), \text{FU}^1(2n, \frac{K}{t^{m'}}, \frac{\Omega}{t^{m'}}) \right] \left[ \text{FU}^1(2n, t^p A, t^p \Lambda), \text{FU}^1(2n, \frac{K}{t^{m'}}, \frac{\Omega}{t^{m'}}) \right].$$

Suppose that  $T_{i,j}(\frac{\gamma}{t^m})$  is a short root. Let  $k \neq \pm i, \pm j$ . Then by (R4), we have

$$T_{i,j}(\frac{\gamma}{t^m}) = [T_{i,k}(t^p), T_{k,j}(\frac{\gamma}{t^{m-p}})],$$

which satisfies our claim.

Suppose that  $T_{i,j}(\frac{\gamma}{t^m})$  is a long root. Let  $k \neq \pm i$ . Using a variation of (R6), we get

$$\begin{aligned} T_{i,-i}(\frac{\gamma}{t^m}) &= T_{i,-i}(t^p \frac{\gamma}{t^{m-2p}} t^p) \\ &= T_{i,-k}(-t^p \lambda^{-(\varepsilon(k)-\varepsilon(i))/2} \frac{\gamma}{t^{m-2p}}) [T_{i,k}(t^p), T_{k,-k}(\lambda^{-(\varepsilon(k)-\varepsilon(i))/2} \frac{\gamma}{t^{m-2p}})]. \end{aligned}$$

By the previous paragraph, we have

$$T_{i,-k}(-\lambda^{-(\varepsilon(k)-\varepsilon(i))/2} \frac{\gamma}{t^{m-p}}) \in \left[ \text{FU}^1(2n, t^p A, t^p \Lambda), \text{FU}^1(2n, \frac{K}{t^{m'}}, \frac{\Omega}{t^{m'}}) \right].$$

This proves the claim.

Together with the identity (C1<sup>+</sup>) and (16) in Lemma 9, the claim above allows us further reduce the proof to show that

$$\begin{aligned} & \left[ [\text{FU}^1(2n, t^p I, t^p \Gamma), \text{FU}^1(2n, J, \Delta)], {}^c [\text{FU}^1(2n, t^p A, t^p \Lambda), \text{FU}^1(2n, \frac{K}{t^{m'}}, \frac{\Omega}{t^{m'}})] \right] \\ & \subseteq \left[ [\text{EU}(2n, t^l I, t^l \Gamma), \text{EU}(2n, t^l J, t^l \Delta)], \text{EU}(2n, t^l K, t^l \Omega) \right]. \end{aligned}$$

Clearly,  $\text{FU}^1(2n, J, \Delta) = \text{FU}^1(2n, \frac{J}{t^0}, \frac{\Delta}{t^0})$ . By (16) in Lemma 9, for any given  $p'$  we have an integer  $p$  such that

$$[\text{FU}^1(2n, t^p I, t^p \Gamma), \text{FU}^1(2n, J, \Delta)] \subseteq [\text{FU}(2n, t^{p'} A, t^{p'} I, t^{p'} \Gamma), \text{FU}(2n, t^{p'} A, t^{p'} J, t^{p'} \Delta)].$$

Therefore, we obtain

$$\begin{aligned}
& \left[ [\text{FU}^1(2n, t^p I, t^p \Gamma), \text{FU}^S(2n, A, J, \Delta)], {}^c [\text{FU}^1(2n, t^p A, t^p \Lambda), \text{FU}^1(2n, \frac{K}{t^{m'}}, \frac{\Omega}{t^{m'}})] \right] \\
& \subseteq \left[ [\text{FU}(2n, t^{p'} A, t^{p'} I, t^{p'} \Gamma), \text{FU}(2n, t^{p'} A, t^{p'} J, t^{p'} \Delta)], \right. \\
& \quad \left. {}^c [\text{FU}^1(2n, t^p A, t^p \Lambda), \text{FU}^1(2n, \frac{K}{t^{m'}}, \frac{\Omega}{t^{m'}})] \right] \\
& \subseteq {}^c \left[ [\text{FU}(2n, t^{p'} A, t^{p'} I, t^{p'} \Gamma), \text{FU}(2n, t^{p'} A, t^{p'} J, t^{p'} \Delta)]^c, \right. \\
& \quad \left. [\text{FU}^1(2n, t^p A, t^p \Lambda), \text{FU}^1(2n, \frac{K}{t^{m'}}, \frac{\Omega}{t^{m'}})] \right].
\end{aligned}$$

Applying (15) in Lemma 9, we find an integer  $p'$  such that

$$\begin{aligned}
& [\text{FU}(2n, t^{p'} A, t^{p'} I, t^{p'} \Gamma), \text{FU}(2n, t^{p'} A, t^{p'} J, t^{p'} \Delta)]^c \\
& \subseteq [\text{FU}(2n, t^{p''} A, t^{p''} I, t^{p''} \Gamma), \text{FU}(2n, t^{p''} A, t^{p''} J, t^{p''} \Delta)]
\end{aligned}$$

for any given integer  $p''$ . Thanks to Lemma 12, for any given  $l'$ , we find an integer  $p''$  such that

$$\begin{aligned}
& [\text{FU}(2n, t^{p''} A, t^{p''} I, t^{p''} \Gamma), \text{FU}(2n, t^{p''} A, t^{p''} J, t^{p''} \Delta)], \\
& [\text{FU}^1(2n, t^p A, t^p \Lambda), \text{FU}^1(2n, \frac{K}{t^{m'}}, \frac{\Omega}{t^{m'}})] \\
& \subseteq {}^c \left[ [\text{FU}(2n, t^{l'} A, t^{l'} I, t^{l'} \Gamma), \text{FU}(2n, t^{l'} A, t^{l'} J, t^{l'} \Delta)], \text{FU}(2n, t^{l'} A, t^{l'} K, t^{l'} \Omega) \right] \\
& = \left[ [{}^c \text{FU}(2n, t^{l'} A, t^{l'} I, t^{l'} \Gamma), {}^c \text{FU}(2n, t^{l'} A, t^{l'} J, t^{l'} \Delta)], {}^c \text{FU}(2n, t^{l'} A, t^{l'} K, t^{l'} \Omega) \right].
\end{aligned}$$

Applying (14) in Lemma 9 to each component of the commutator above, we may find a sufficiently large integer  $l'$  such that for any given  $l$

$$\begin{aligned}
& \left[ [{}^c \text{FU}(2n, t^{l'} A, t^{l'} I, t^{l'} \Gamma), {}^c \text{FU}(2n, t^{l'} A, t^{l'} J, t^{l'} \Delta)], {}^c \text{FU}(2n, t^{l'} A, t^{l'} K, t^{l'} \Omega) \right] \\
& \subseteq \left[ [\text{EU}(2n, t^l I, t^l \Gamma), \text{EU}(2n, t^l J, t^l \Delta)], \text{EU}(2n, t^l K, t^l \Omega) \right].
\end{aligned}$$

Hence,

$$[e_1, e_2] \in \left[ [\text{EU}(2n, t^l I, t^l \Gamma), \text{EU}(2n, t^l J, t^l \Delta)], \text{EU}(2n, t^l K, t^l \Omega) \right],$$

which finishes the proof.  $\square$

## 9. PROOF OF THEOREM 7

Now we are all set to complete the proof of the *triple* commutator formula, Theorem 7.

The functors  $\text{EU}_{2n}$  and  $\text{GU}_{2n}$  commute with direct limits. By §4.3, one reduces the proof to the case where  $A$  is finite over  $R_0$  and  $R_0$  is Noetherian.

By the relative standard commutator formula, Theorem 4, we have

$$[\text{EU}(2n, I, \Gamma), \text{GU}(2n, J, \Delta)] = [\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)].$$

Thus, to prove Theorem 7 it suffices to prove the following equality

$$\left[ [\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)], \text{GU}(2n, K, \Omega) \right] = \left[ [\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)], \text{EU}(2n, K, \Omega) \right].$$

By Theorem 9 the mixed commutator subgroup  $[\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)]$  is generated by the conjugates in  $\text{EU}(2n, A, \Lambda)$  of the following types of elements

$$e = [T_{ji}(\alpha), {}^{T_{ij}(a)}T_{ji}(\beta)], \quad e = [T_{ji}(\alpha), T_{ij}(\beta)], \quad \text{and} \quad e = T_{ij}(\xi), \quad (19)$$

where  $\alpha \in (I, \Gamma)$ ,  $\beta \in (J, \Delta)$ ,  $\xi \in (I, \Gamma) \circ (J, \Delta)$  and  $a \in (A, \Lambda)$ .

We claim that for any  $g \in \text{GU}(2n, K, \Omega)$ ,

$$[e, g] \in \left[ [\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)], \text{EU}(2n, K, \Omega) \right]. \quad (20)$$

Let  $g \in \text{GU}(2n, K, \Omega)$ . For any maximal ideal  $\mathfrak{m} \in \text{Max}(R_0)$ , the form ring  $(A_{\mathfrak{m}}, \Lambda_{\mathfrak{m}})$  contains  $(K_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$  as a form ideal. Consider the localisation homomorphism  $F_{\mathfrak{m}} : A \rightarrow A_{\mathfrak{m}}$  which induces homomorphisms on the level of unitary groups,

$$F_{\mathfrak{m}} : \text{GU}(2n, A, \Lambda) \rightarrow \text{GU}(2n, A_{\mathfrak{m}}, \Lambda_{\mathfrak{m}}),$$

and

$$F_{\mathfrak{m}} : \text{GU}(2n, K, \Omega) \rightarrow \text{GU}(2n, K_{\mathfrak{m}}, \Omega_{\mathfrak{m}}).$$

Therefore, for  $g \in \text{GU}(2n, K, \Omega)$ ,  $F_{\mathfrak{m}}(g) \in \text{GU}(2n, K_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$ . Since  $A_{\mathfrak{m}}$  is module finite over the local ring  $R_{\mathfrak{m}}$ ,  $A_{\mathfrak{m}}$  is semi-local [11, III(2.5), (2.11)], therefore its stable rank is 1. It follows by (see [19, 9.1.4]) that,

$$\text{GU}(2n, K_{\mathfrak{m}}, \Omega_{\mathfrak{m}}) = \text{EU}(2n, K_{\mathfrak{m}}, \Omega_{\mathfrak{m}}) \text{GU}(2, K_{\mathfrak{m}}, \Omega_{\mathfrak{m}}).$$

Thus,  $F_{\mathfrak{m}}(g)$  can be decomposed as  $F_{\mathfrak{m}}(g) = \varepsilon h$ , where  $\varepsilon \in \text{EU}(2n, K_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$  and  $h \in \text{GU}(2, K_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$  is a  $2 \times 2$  matrix embedded in  $\text{GU}(2n, K_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$  and this embedding can be arranged modulo  $\text{EU}(2n, K_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$ .

Now, by (4.3), we may reduce the problem to the case  $A_t$  with  $t \in R_0 \setminus \mathfrak{m}$ . Namely,

$$F_t(g) = \varepsilon h, \quad (21)$$

where  $\varepsilon \in \text{EU}(2n, K_t, \Omega_t)$  and  $h \in \text{GL}(2, K_t, \Omega_t)$ .

For any maximal ideal  $\mathfrak{m} \triangleleft R_0$ , choose  $t_{\mathfrak{m}} \in R_0 \setminus \mathfrak{m}$  as above and an arbitrary positive integer  $p_{\mathfrak{m}}$ . (We will later choose  $p_{\mathfrak{m}}$  according to Lemma 13.) Since the collection of all  $\{t_{\mathfrak{m}}^{p_{\mathfrak{m}}} \mid \mathfrak{m} \in \text{max}(R_0)\}$  is not contained in any maximal ideal, we may find a finite number of  $t_{\mathfrak{m}_s}^{p_s} \in R_0 \setminus \mathfrak{m}_s$  and  $x_s \in R_0$ ,  $s = 1, \dots, k$ , such that

$$\sum_{s=1}^k t_{\mathfrak{m}_s}^{p_s} x_s = 1.$$

In order to prove (20), first we consider the generators of the first kind in (19), namely

$$e = [T_{ji}(\alpha), {}^{T_{ij}(a)}T_{ji}(\beta)].$$

Consider

$$e = [T_{ji}(\alpha), {}^{T_{ij}(a)}T_{ji}(\beta)] = \left[ T_{ji} \left( \sum_{s=1}^k t_{\mathfrak{m}_s}^{p_s} x_s \alpha \right), {}^{T_{ij}(a)}T_{ji}(\beta) \right] = \left[ \prod_{s=1}^k T_{ji}(t_{\mathfrak{m}_s}^{p_s} x_s \alpha), {}^{T_{ij}(a)}T_{ji}(\beta) \right].$$

By Identity (C2<sup>+</sup>), the element  $e = \left[ \prod_{s=1}^k T_{ji}(t_{\mathfrak{m}_s}^{p_s} x_s \alpha), {}^{T_{ij}(a)}T_{ji}(\beta) \right]$  can be written as a product of the following form:

$$e = T_k \left[ T_{ji}(t_{\mathfrak{m}_k}^{p_k} x_k \alpha), {}^{T_{ij}(a)}T_{ji}(\beta) \right] \cdot T_{k-1} \left[ T_{ji}(t_{\mathfrak{m}_{k-1}}^{p_{k-1}} x_{k-1} \alpha), {}^{T_{ij}(a)}T_{ji}(\beta) \right] \cdot \dots \cdot T_1 \left[ T_{ji}(t_{\mathfrak{m}_1}^{p_1} x_1 \alpha), {}^{T_{ij}(a)}T_{ji}(\beta) \right], \quad (22)$$

where  $T_1, T_2, \dots, T_k \in \text{EU}(2n, A, \Lambda)$ . Note that from (C2<sup>+</sup>) it is clear that all  $T_s$ ,  $s = 1, \dots, k$ , are products of elementary matrices of the form  $T_{ji}(a)$ . Thus  $T_s = T_{ji}(a_s)$ , where  $a_s \in A$  and  $s = 1, \dots, k$ , which clearly commutes with  $T_{ji}(x)$  for any  $x \in A$ . So the commutator (22) is equal to

$$e = \left[ T_{ji}(t_{\mathfrak{m}_k}^{p_k} x_k \alpha), {}^{T_k T_{ij}(a)}T_{ji}(\beta) \right] \cdot \left[ T_{ji}(t_{\mathfrak{m}_{k-1}}^{p_{k-1}} x_{k-1} \alpha), {}^{T_{k-1} T_{ij}(a)}T_{ji}(\beta) \right] \cdot \dots \cdot \left[ T_{ji}(t_{\mathfrak{m}_1}^{p_1} x_1 \alpha), {}^{T_1 T_{ij}(a)}T_{ji}(\beta) \right]. \quad (23)$$

Using (C2<sup>+</sup>) and in view of (23) we obtain that  $[e, g]$  is a product of the conjugates in  $\text{EU}(2n, A, \Lambda)$  of

$$w_s = \left[ \left[ T_{ji}(t_{\mathfrak{m}_s}^{p_s} x_s \alpha), {}^{T_{ji}(a_s) T_{ij}(a)}T_{ji}(\beta) \right], g \right],$$

where  $a_s \in A$  and  $s = 1, \dots, k$ .

For each  $s = 1, \dots, k$ , consider  $\theta_{t_{\mathfrak{m}_s}}(w_s)$  which we still write as  $w_s$  but keep in mind that this image is in  $\text{GU}(2n, A_{t_{\mathfrak{m}_s}}, \Lambda_{t_{\mathfrak{m}_s}})$ .

Note that all  $\left[ T_{ji}(t_{\mathfrak{m}_s}^{p_s} x_s \alpha), {}^{T_{ji}(a_s) T_{ij}(a)}T_{ji}(\beta) \right]$ ,  $s = 1, \dots, k$ , differ from the identity matrix at only the rows  $\pm i, \pm j$  and in the columns  $\pm i, \pm j$ . Since  $n > 2$ , we can choose an  $h$  in the decomposition (21) so that it commutes with

$$\left[ T_{ji}(t_{\mathfrak{m}_s}^{p_s} x_s \alpha), {}^{T_{ji}(a_s) T_{ij}(a)}T_{ji}(\beta) \right].$$

This allows us to reduce  $\theta_{t_{\mathfrak{m}_s}}(w_s)$  to

$$\left[ \left[ T_{ji}(t_{\mathfrak{m}_s}^{p_s} x_s \alpha), {}^{T_{ji}(a_s) T_{ij}(a)}T_{ji}(\beta) \right], \varepsilon \right],$$

where  $\varepsilon \in E_n(A_{t_{m_s}}, K_{t_{m_s}})$ . By Lemma 13, for any given  $l_s$ , there is a sufficiently large  $p_s$ ,  $s = 1, \dots, k$ , such that

$$\left[ \left[ T_{ji}(t_{m_s}^{p_s} x_s \alpha), T_{ji}(a_s) T_{ij}(a) T_{ji}(\beta) \right], \varepsilon \right] \in \left[ \left[ \text{EU}(2n, t^{l_s} I, t^{l_s} \Gamma), \text{EU}(2n, t^{l_s} J, t^{l_s} \Delta) \right], \text{EU}(2n, t^{l_s} K, t^{l_s} \Omega) \right].$$

Let us choose  $l_s$  to be large enough so that by Lemma 6 the restriction of

$$\theta_{t_{m_s}} : \text{GL}_n(A, t_{m_s}^{l_s} A) \rightarrow \text{GL}_n(A_{t_{m_s}})$$

be injective. Then it is easy to see that for any  $s$ , we have

$$\left[ \left[ T_{ji}(t_{m_s}^{p_s} x_s \alpha), T_{ji}(a_s) T_{ij}(a) T_{ji}(\beta) \right], g \right] \in \left[ \left[ \text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta) \right], \text{EU}(2n, K, \Omega) \right].$$

Since relative elementary subgroups are normal in  $\text{GU}(2n, A, \Lambda)$  (Theorem 2), it follows that

$$[e, g] \in \left[ \left[ \text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta) \right], \text{EU}(2n, K, \Omega) \right].$$

When the generator is of the second kind,  $e = [T_{ij}(\alpha), T_{ji}(\beta)]$ , a similar argument goes through, which is left to the reader.

Now consider the generators of the 3rd kind, namely, the conjugates of the following type of elements,  $e = T_{ij}(\alpha\beta)$ . By the normality of  $\text{EU}(2n, (I, \Gamma) \circ (J, \Delta))$ , the conjugates of  $e$  are in  $\text{EU}(2n, (I, \Gamma) \circ (J, \Delta))$ . We have

$$[e, g] \in \left[ \text{EU}(2n, (I, \Gamma) \circ (J, \Delta)), \text{GU}(2n, K, \Omega) \right].$$

By the generalized commutator formula (Theorem 4), one obtains

$$\left[ \text{EU}(2n, (I, \Gamma) \circ (J, \Delta)), \text{GU}(2n, K, \Omega) \right] = \left[ \text{EU}(2n, (I, \Gamma) \circ (J, \Delta)), \text{EU}(2n, K, \Omega) \right].$$

Now applying Lemma 7, we finally get

$$\left[ \text{EU}(2n, (I, \Gamma) \circ (J, \Delta)), \text{EU}(2n, K, \Omega) \right] \subseteq \left[ \left[ \text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta) \right], \text{EU}(2n, K, \Omega) \right].$$

Therefore,  $[e, g] \in \left[ \left[ \text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta) \right], \text{EU}(2n, K, \Omega) \right]$ . This proves our claim. Thus we established (20) for all type of generators  $e$  of (19).

To finish the proof, let  $e \in [\text{EU}(2n, I, \Gamma), \text{GU}(2n, J, \Delta)] = [\text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta)]$ , and  $g \in \text{GU}(2n, K, \Omega)$ . Then by Lemma 9,

$$e = c_1 e_1 \times c_2 e_2 \times \dots \times c_k e_k,$$

with  $c_i \in \text{EU}(2n, A, \Lambda)$  and  $e_i$  takes any of the forms in (19). Since the relative elementary subgroups are normal, Identity (C2<sup>+</sup>) implies that it suffices to show that

$$[c_i e_i, g] \in \left[ \left[ \text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta) \right], \text{EU}(2n, K, \Omega) \right], \quad i = 1, \dots, k.$$

Now, since both the relative elementary subgroups and the principal congruence subgroups are normal, Identity (C5) further reduces the problem to verification of the inclusions

$$[e_i, g] \in \left[ \left[ \text{EU}(2n, I, \Gamma), \text{EU}(2n, J, \Delta) \right], \text{EU}(2n, K, \Omega) \right], \quad i = 1, \dots, k.$$

But this is exactly what has been shown above. This completes the proof of Theorem 7, the rest is now an exercise.

## 10. MULTIPLE COMMUTATOR FORMULAS FOR GROUP FUNCTORS

Here we finish the proof of Theorems 5 and 6. In fact, we verify that these theorems formally follow from Theorems 4 and 7 and Lemmas 7 and 8.

Namely, let  $G_0, \dots, G_n$ ,  $n \geq 3$ , be subgroups of a given group  $G$ . There are many ways to arrange brackets  $[\_, \_]$  in the sequence  $G_0, \dots, G_n$  to correctly define the multi-commutator of these subgroups. For instance, for  $n = 4$ , we can have the following two arrangements  $\left[[G_0, [G_1, G_2]], G_3\right]$  and  $\left[[G_0, G_1], [G_2, G_3]\right]$ , among others. It is classically known that overall there are

$$c_n = \frac{1}{(n+1)} \binom{2n}{n}$$

ways to arrange brackets to form a multi-commutator of  $n+1$  subgroups, where  $c_n$  is the Catalan number. Any such arrangement correctly defining a multi-commutator will be denoted by

$$\llbracket G_0, G_1, \dots, G_m \rrbracket.$$

This notation is introduced to distinguish general such arrangements from the standard multi-commutator  $[G_0, \dots, G_n]$ , which is usually interpreted as the left-normed commutator,

$$[G_0, \dots, G_n] = \left[ \dots \left[ [G_0, G_1], G_2 \right], \dots, G_n \right].$$

Let us fix the axiomatic setting for the proof of Theorems 5 and 6. Let  $A$  be a ring. For each two-sided ideal  $I$  of  $A$ , let  $E(I)$  and  $G(I)$  be subgroups of  $G = G(A)$  such that  $E(I)$  is a normal subgroup of  $G(I)$ . Assume, for any three two-sided ideals  $I, J$  and  $K$  of  $A$  the following holds

- (M1)  $E(I) \subseteq E(J)$  and  $G(I) \subseteq G(J)$ ,
- (M2)  $[E(I), G(J)] = [E(I), E(J)]$ ,
- (M3)  $\left[[E(I), G(J)], G(K)\right] = \left[[E(I), E(J)], E(K)\right]$ ,
- (M4)  $E(I \circ J) \subseteq [E(I), E(J)] \subseteq [E(I), G(J)] \subseteq [G(I), G(J)] \subseteq G(I \circ J)$ .

Recall, that here we denote by  $I \circ J = IJ + JI$  the symmetrised product of the ideals  $I$  and  $J$ . This operation is not associative, so when writing  $I_0 \circ \dots \circ I_m$  we assume that this is the left-normed product. In particular,  $I \circ J \circ K = (I \circ J) \circ K$ .

*Example 14.* Let  $A$  be a quasi-finite  $R$ -algebra. The main results of [32] show that for any two-sided ideal  $I$  of  $A$ ,  $E(I) = E_n(A, I)$  and  $G(I) = \text{GL}_n(A, I)$  satisfy Conditions (M1)–(M4).

*Example 15.* Let  $(A, \Lambda)$  be a quasi-finite form ring and let  $E(I_i) = \text{EU}(2n, I_i, \Gamma_i)$  and  $G(I_i) = \text{GU}(2n, I_i, \Gamma_i)$ . Then Lemma 7, Theorem 4, Lemma 8 and Theorem 7 show that Conditions (M1)–(M4) are satisfied in this setting.

The following lemma proves Theorem 5.

**Lemma 16.** *Let  $A$  be an  $R$ -algebra,  $I_i$ ,  $i = 0, \dots, m$ , be two-sided ideals of  $A$ . Assume that subgroups  $G(I)$  and  $E(I)$  satisfy Conditions (M1)–(M4). Then*

$$[E(I_0), G(I_1), G(I_2), \dots, G(I_m)] = [E(I_0), E(I_1), E(I_2), \dots, E(I_m)]. \quad (24)$$

*Proof.* The proof proceeds by induction on  $m$ . For  $i = 1$  this is Condition (M2). For  $i = 2$ , this is Condition (M3) which will be the first step of induction. Suppose the statement is valid for  $m - 1$ , when there are  $m$  ideals in the commutator formula. By Condition (M3), we have

$$\left[ \left[ [E(I_0), G(I_1)], G(I_2) \right], G(I_3), \dots, G(I_m) \right] = \left[ \left[ [E(I_0), E(I_1)], E(I_2) \right], G(I_3), \dots, G(I_m) \right].$$

On the other hand, by Condition (M4) one has  $[E(I_0), E(I_1)] \subseteq G(I_0 I_1 + I_1 I_0)$ . Thus

$$\left[ \left[ [E(I_0), E(I_1)], E(I_2) \right], G(I_3), \dots, G(I_m) \right] \subseteq \left[ \left[ G(I_0 I_1 + I_1 I_0), E(I_2) \right], G(I_3), \dots, G(I_m) \right].$$

Since there are  $m$  ideals involved in the commutator subgroups on the right hand side, we can apply the induction hypothesis and get

$$\left[ \left[ G(I_0 I_1 + I_1 I_0), E(I_2) \right], G(I_3), \dots, G(I_m) \right] = \left[ \left[ E(I_0 I_1 + I_1 I_0), E(I_2) \right], E(I_3), \dots, E(I_m) \right].$$

Finally, invoking Condition (M4) once more, we get  $E(I_0 I_1 + I_1 I_0) \subseteq [E(I_0), E(I_1)]$ . Substituting this inclusion in the above equality we can conclude that the left hand side of (24) is contained in the right hand side. Since the opposite inclusion is obvious, this completes the proof.  $\square$

Now, we can go one step further, and show that in fact it does not matter where the elementary subgroup appears in the multiple commutator formula.

**Lemma 17.** *Let  $A$  be an  $R$ -algebra and  $I_i$ ,  $i = 0, \dots, m$ , be two-sided ideals of  $A$ . Assume that subgroups  $G(I)$  and  $E(I)$  satisfy Conditions (M1)–(M4). Let  $G_i$  be subgroups of  $G(A)$  such that*

$$E(I_i) \subseteq G_i \subseteq G(I_i), \quad \text{for } i = 0, \dots, m.$$

*If there is an index  $j$  such that  $G_j = E(I_j)$ , then*

$$[G_0, G_1, \dots, G_m] = [E(I_0), E(I_1), E(I_2), \dots, E(I_m)]. \quad (25)$$

*Proof.* For brevity, denote  $E(I_i)$  by  $E_i$ . For a fixed  $m$  the proof proceeds on induction on  $j$ . As the base of induction one takes  $j = 0$ , which is the previous lemma. When  $j = 1$  one has  $[G_0, E_1] = [E_1, G_0]$ , so this case reduces to the case of  $j = 0$ .

For  $2 \leq j \leq m$  we can argue as follows. By assumption

$$\begin{aligned} [G_0, G_1, \dots, G_j, G_{j+1}, \dots, G_m] &= [[G_0, G_1, \dots, G_j], G_{j+1}, \dots, G_m] = \\ &= [[G_0, G_1, \dots, G_{j-1}, E_j], G_{j+1}, \dots, G_m] = [[G_0, G_1, \dots, G_{j-1}], E_j, G_{j+1}, \dots, G_m]. \end{aligned}$$

Now, repeated application of the rightmost inclusion from Condition (M4) shows that

$$[G_0, G_1, \dots, G_{k-1}] \subseteq G(I_0 \circ \dots \circ I_{k-1}).$$

Combining Conditions (M2) and (M4), we get

$$\begin{aligned} [E_0, E_1, \dots, E_{k-1}, E_k] &\subseteq [[G_0, G_1, \dots, G_{k-1}], E_k] \\ &\subseteq [G(I_0 \circ \dots \circ I_{k-1}), E_k] = [E(I_0 \circ \dots \circ I_{k-1}), E_k] \subseteq [[E_0, E_1, \dots, E_{k-1}], E_k], \end{aligned}$$

and thus

$$[G_0, G_1, \dots, G_{k-1}, E_k] = [[E_0, E_1, \dots, E_{k-1}], E_k].$$

Substituting this into our commutator, we see that

$$[G_0, G_1, \dots, G_m] = [E_0, E_1, \dots, E_{k-1}, E_k, G_{k+1}, \dots, G_m],$$

and it only remains to invoke the previous lemma.  $\square$

Now we are all set for the final round of computation, to show that it does not matter how the brackets are arranged either. In particular, this proves Theorem 6.

**Lemma 18.** *Let  $A$  be an  $R$ -algebra and  $I_i$ ,  $i = 0, \dots, m$ , be two-sided ideals of  $A$ . Assume that subgroups  $G(I)$  and  $E(I)$  satisfy Conditions (M1)–(M4). Let  $G_i$  be subgroups of  $G(A)$  such that*

$$E(I_i) \subseteq G_i \subseteq G(I_i), \quad \text{for } i = 0, \dots, m.$$

*If there is an index  $j$  such that  $G_j = E(I_j)$ , then*

$$[[G_0, G_1, \dots, G_m]] = [[E(I_0), E(I_1), \dots, E(I_m)]]. \quad (26)$$

*Proof.* To prove (26), we proceed by induction on  $m$ . For  $m = 0$  and  $m = 1$  there is nothing to prove. For  $m = 2$ , the commutator  $[[G_0, G_1, G_2]]$  can be arranged in six possible ways

$$\begin{aligned} & [[G_0, G_1], E_2], \quad [E_0, [G_1, G_2]], \quad [[E_0, G_1], G_2], \\ & \quad \quad \quad [[G_0, E_1], G_2], \quad [G_0, [E_1, G_2]], \quad [G_0, [G_1, E_2]], \end{aligned}$$

of which the first two and the last four are reduced to each other by (C6), the commutativity of commutator on subgroups. Thus, it only matters, whether  $E_j$  stands inside the inner bracket, or outside of it. The case, where it stands inside, was already considered in the previous lemma, so that it only remains to consider the first of the above arrangements. Using Conditions (M1)–(M4) we get

$$\begin{aligned} [[E_0, E_1], E_2] & \subseteq [[G_0, G_1], E_2] \subseteq [[G(I_0), G(I_1)], E(I_2)] \\ & \subseteq [G(I_0 \circ I_1), E(I_2)] = [E(I_0 \circ I_1), E(I_2)] \\ & \subseteq [[E(I_0), E(I_1)], E(I_2)] = [[E_0, E_1], E_2], \end{aligned}$$

as claimed.

For the main step of induction, we consider two cases. Suppose first there is a mixed commutator  $[G_i, G_{i+1}]$  in  $[[G_0, G_1, \dots, G_m]]$ , where neither  $G_i$  nor  $G_{i+1}$  is the *fixed* elementary subgroup  $E_j$ . Then

$$\begin{aligned} [[G_0, G_1, \dots, G_m]] & = [[G_0, G_1, \dots, [G_i, G_{i+1}], \dots, G_m]] \\ & \subseteq [[G_0, G_1, \dots, [G(I_i), G(I_{i+1})], \dots, G_m]] \\ & \subseteq [[G_0, G_1, \dots, G(I_i I_{i+1} + I_{i+1} I_i), \dots, G_m]]. \end{aligned} \quad (27)$$

Note that there is one fewer ideal involved in the last commutator formula (i.e.,  $m - 1$  ideals) which also contains an elementary subgroup, and so by induction

$$\begin{aligned}
 \llbracket G_0, G_1, \dots, G(I_i I_{i+1} + I_{i+1} I_i), \dots, G_m \rrbracket \\
 &= \llbracket E_0, E_1, \dots, E(I_i I_{i+1} + I_{i+1} I_i), \dots, E_m \rrbracket \\
 &\subseteq \llbracket E_0, E_1, \dots, [E(I_i), E(I_{i+1})], \dots, E_m \rrbracket \\
 &= \llbracket E_0, E_1, \dots, E_m \rrbracket.
 \end{aligned} \tag{28}$$

Putting 27 and 28 together, we get

$$\llbracket G_0, G_1, \dots, G_m \rrbracket = \llbracket E_0, E_1, \dots, E_m \rrbracket.$$

It only remains to consider the case, where the only double mixed commutator of the form  $[G_i, G_{i+1}]$  inside our arrangement  $\llbracket G_0, G_1, \dots, G_m \rrbracket$  involves our fixed elementary subgroup  $E_j$ . Consider the outermost pairs of inner brackets in our multicommutator

$$\llbracket G_0, G_1, \dots, G_m \rrbracket = \left[ \llbracket G_0, G_1, \dots, G_k \rrbracket, \llbracket G_{k+1}, \dots, G_m \rrbracket \right].$$

If  $1 \leq k \leq m-2$ , then *each* of the inner brackets  $\llbracket G_0, G_1, \dots, G_k \rrbracket$  and  $\llbracket G_{k+1}, \dots, G_m \rrbracket$  contains a double mixed commutator, one of which leaves  $E_j$  outside, and this is the situation we just considered.

This leaves us with the analysis of the case, where  $k = 0$  or  $k = m - 1$ , in other words,  $\llbracket G_0, G_1, \dots, G_m \rrbracket$  is one of the following

$$\left[ G_0, \llbracket G_1, \dots, G_m \rrbracket \right], \quad \left[ \llbracket G_0, G_1, \dots, G_{m-1} \rrbracket, G_m \right].$$

By commutativity of the commutator, the first of these situations reduces to the second one.

Repeating this argument, we see that – modulo the commutativity of the commutator – the arrangement  $\llbracket G_0, G_1, \dots, G_m \rrbracket$  is the left-normed one, for which the previous lemma already guarantees that

$$\llbracket G_0, G_1, \dots, G_m \rrbracket = [G_{i_0}, G_{i_1}, \dots, G_{i_m}] = [E_{i_0}, E_{i_1}, \dots, E_{i_m}] = \llbracket E_0, E_1, \dots, E_m \rrbracket.$$

This finishes the proof of Lemma 18 and thus also of Theorem 6.  $\square$

Our final lemma proves Theorem 8. Recall that the product of ideals are not associative. Thus in the following Lemma the bracketings of the form ideals on the right hand side should correspond to the bracketings of commutators on the left-hand side.

**Lemma 19.** *Let  $A$  be an  $R$ -algebra and  $I_i$ ,  $i = 0, \dots, m$ , be two-sided ideals of  $A$ . Assume that subgroups  $G(I)$  and  $E(I)$  satisfy Conditions (M1)–(M4). Then*

$$\begin{aligned}
 \left[ \llbracket E(I_0), E(I_1), \dots, E(I_k) \rrbracket, \llbracket E(I_{k+1}), \dots, E(I_m) \rrbracket \right] = \\
 [E(I_0 \circ \dots \circ I_k), E(I_{k+1} \circ \dots \circ I_m)],
 \end{aligned}$$

where the bracketing of symmetrised products on the right hand side coincides with the bracketing of the commutators on the left hand side.

*Proof.* Alternated application of (M4) and (M2) shows that

$$\begin{aligned} & \left[ \llbracket E(I_0), E(I_1), \dots, E(I_k) \rrbracket, \llbracket E(I_{k+1}), \dots, E(I_m) \rrbracket \right] \leq \\ & \left[ G(I_0 \circ \dots \circ I_k), \llbracket E(I_{k+1}), \dots, E(I_m) \rrbracket \right] = \left[ E(I_0 \circ \dots \circ I_k), \llbracket E(I_{k+1}), \dots, E(I_m) \rrbracket \right] \leq \\ & \left[ E(I_0 \circ \dots \circ I_k), G(I_{k+1} \circ \dots \circ I_m) \right] = \left[ E(I_0 \circ \dots \circ I_k), E(I_{k+1} \circ \dots \circ I_m) \right] \leq \\ & \left[ \llbracket E(I_0), E(I_1), \dots, E(I_k) \rrbracket, \llbracket E(I_{k+1}), \dots, E(I_m) \rrbracket \right], \end{aligned}$$

as claimed.  $\square$

## 11. FINAL REMARKS

The present paper grew out of desire to prove the *general* multiple commutator formula, which simultaneously generalises our multiple commutator formula and nilpotent filtration of relative  $K_1$ , see [5]. Of course, the general commutator formula can only hold for finite-dimensional rings. It can be stated as follows.

**Problem 1.** *Let  $R$  be a ring of finite Bass–Serre dimension  $\delta(R) = d < \infty$ , and let  $(I_i, \Gamma_i)$ ,  $1 \leq i \leq m$ , be form ideals of  $(R, \Lambda)$ . Prove that for any  $m \geq d$  one has*

$$\begin{aligned} & \left[ \mathrm{GU}(2n, I_0, \Gamma_0), \mathrm{GU}(2n, I_1, \Gamma_1), \mathrm{GU}(2n, I_2, \Gamma_2), \dots, \mathrm{GU}(2n, I_m, \Gamma_m) \right] = \\ & \left[ \mathrm{EU}(2n, I_0, \Gamma_0), \mathrm{EU}(2n, I_1, \Gamma_1), \mathrm{EU}(2n, I_2, \Gamma_2), \dots, \mathrm{EU}(2n, I_m, \Gamma_m) \right]. \end{aligned}$$

In fact, recently we succeeded in proving such a formula for general linear groups [25]. However the proof there critically depends on a number of deep external results. In this respect the case of unitary groups seems to be very different, since in the context of unitary groups even the most basic results are simply not there in the existing literature.

For instance, the proof in [25] starts with the following classical observation by Alec Mason and Wilson Stothers [37], which serves as the base of induction.

**Theorem 10** (Mason–Stothers). *Let  $R$  be a ring,  $I$  and  $J$  be two two-sided ideals of  $R$ . Assume that  $n \geq \mathrm{sr}(R), 3$ . Then*

$$\left[ \mathrm{GL}(n, R, I), \mathrm{GL}(n, R, J) \right] = \left[ E(n, R, I), E(n, R, J) \right].$$

For unitary groups, even such basic facts at the stable level seem to be missing. After that the proof in [25] proceeds by induction on  $d$ , which depends on Bak’s results [3], precise form of injective stability for  $K_1$ , such as the Bass–Vaserstein theorem, etc. It seems that to solve Problem 1 one has to rethink and expand many aspects of structure theory of unitary groups, starting with stability theorems [7, 6, 45], more powerful analogues of results on the superspecial unitary groups, than what we established in [20, 21, 5], etc. All these results seem feasible, but to actually set them afoot might take a lot of work.

Let us mention two further problems closely related to the contents of the present paper. Firstly, multiple commutator formulas are relevant for the description of subnormal subgroups of  $\mathrm{GU}(2n, A, \Lambda)$ . Important progress in this direction was recently obtained by the third author and You Hong [65, 63]. But we feel that the bounds in these results can be improved and hope to return to this problem with our new tools.

Secondly, the generators constructed in Theorem 9 are a first approximation to the “elementary” generators of the double commutator subgroups  $[\mathrm{EU}(2n, I, \Gamma), \mathrm{EU}(2n, J, \Delta)]$ .

Actually, building upon this theorem, we constructed a much smaller set of generators, using which Alexei Stepanov was able to prove finiteness results for relative commutators, see [23, 24].

We refer the interested reader to our forthcoming papers [23, 24, 25, 26], where these and some other related problems are discussed in somewhat more detail.

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